Rigorous || Structured || Proof-Oriented

Refoma Core | Foundations of Mathematics | 2026

ALGEBER ONE

This book presents the basic principles of algebra, focusing on sets, mappings, and algebraic structures. The content is organized progressively, with examples and exercises designed to strengthen conceptual understanding and algebraic reasoning.

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Algebra I

Refoma Core — Algebra

Course Summary & Solved Exercises

Subject: Algebra I
Educational level: 1st Year University mathematics
Student: Abdeljalil Mostefaoui

Math Note:
"Every equation reminds us that nothing in creation is random."

Designed to support students in mastering the fundamental concepts of Algebra.

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First Edition

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Typeset in LaTeX.

Dedication

All praise and gratitude are due to Allah, the Almighty, whose guidance, mercy, and blessings have made this work possible. Through His wisdom and support, I was granted the strength, patience, and determination required to pursue knowledge and to complete this academic journey.
I dedicate this work with profound respect, love, and gratitude to my parents, who have been the cornerstone of my education. To my beloved father, **Abdelhamid Mostefaoui**, whose unwavering dedication, sacrifices, and moral guidance have always inspired me to strive for excellence. His constant encouragement, wisdom, and belief in the value of education have shaped my character and motivated me to persevere through challenges.
To my dear mother, **Mebarka.W.Z**, whose endless love, patience, and prayers have been my greatest source of strength. Her quiet sacrifices, sincere support, and unconditional care have illuminated my path and provided me with the confidence and serenity necessary to pursue my studies with determination and humility. Words cannot fully express my gratitude for her role in every achievement I reach.
Abdeljalil Mostefaoui

Contents

1 Basic Concepts of Logic **6**
1.1 Comprehensive Course Summary . . . 6
1.1.1 1. Fundamental Logical Connectives . . . 6
1.1.2 2. The Art of Quantifiers (Deep Understanding) . . . 6
1.1.3 3. Standard Proof Strategies (Methodology) . . . 6
1.1.4 4. Negation Rules (De Morgan for Quantifiers) . . . 7
1.2 List of Exercises . . . 8
1.2.1 Exercise 01 . . . 8
1.2.2 Exercise 02 . . . 8
1.2.3 Exercise 03 . . . 9
1.2.4 Exercise 04 . . . 9
1.2.5 Exercise 05 . . . 10
1.3 Detailed Solutions and Proofs . . . 11
1.3.1 Solution 01 . . . 11
1.3.2 Solution 02 . . . 13
1.3.3 Solution 03 . . . 14
1.3.4 Solution 04 . . . 14
1.3.5 Solution 05 . . . 16

2 Sets and Mappings **17**
2.1 Comprehensive Course Summary . . . 17
2.1.1 1. Proof Strategies (Methodology) . . . 17
2.1.2 2. Cheat Sheet: Definitions & Operations . . . 17
2.1.3 3. Important Notes for Proofs . . . 18
2.2 List of Exercises . . . 19
2.2.1 Exercise 01 . . . 19
2.2.2 Exercise 02 . . . 19
2.2.3 Exercise 03 . . . 19
2.2.4 Exercise 04 . . . 20
2.2.5 Exercise 05 . . . 20
2.2.6 Exercise 06 . . . 20
2.2.7 Exercise 07 . . . 21
2.2.8 Exercise 08 . . . 21
2.2.9 Exercise 09 . . . 21
2.2.10 Exercise 10 . . . 22
2.2.11 Exercise 11 . . . 22
2.3 Detailed Solutions and Proofs . . . 23
2.3.1 Solution 01 . . . 23
2.3.2 Solution 02 . . . 25
2.3.3 Solution 03 . . . 26
2.3.4 Solution 04 . . . 27
2.3.5 Solution 05 . . . 28
2.3.6 Solution 06 . . . 29
2.3.7 Solution 07 . . . 30
2.3.8 Solution 08 . . . 30
2.3.9 Solution 09 . . . 31
2.3.10 Solution 10 . . . 32
2.3.11 Solution 11 . . . 33

3 Binary Relations on a Set **34**
3.1 Comprehensive Course Summary . . . 34
3.1.1 1. The Four Fundamental Properties . . . 34
3.1.2 2. Types of Relations . . . 34
3.1.3 3. How to Write Rigorous Proofs (Methodology) . . . 35
3.2 List of Exercises . . . 36
3.2.1 Exercise 01 . . . 36
3.2.2 Exercise 02 . . . 36
3.2.3 Exercise 03 . . . 37
3.2.4 Exercise 04 . . . 37
3.2.5 Exercise 05 . . . 37
3.3 Detailed Solutions and Proofs . . . 38
3.3.1 Solution 01 . . . 38
3.3.2 Solution 02 . . . 39
3.3.3 Solution 03 . . . 40
3.3.4 Solution 04 . . . 41
3.3.5 Solution 05 . . . 42

4 Algebraic Structures **43**
4.1 Comprehensive Course Summary . . . 43
4.1.1 Algebraic Structures Summary . . . 43
4.1.2 Keys to Solving Exercises (Moyens & Astuces) . . . 44
4.1.3 Visual Illustrations (Schémas) . . . 44
4.2 List of Exercises . . . 45
4.2.1 Exercise 01 . . . 45
4.2.2 Exercise 02 . . . 45
4.2.3 Exercise 03 . . . 45
4.2.4 Exercise 04 . . . 46
4.2.5 Exercise 05 . . . 46
4.3 Detailed Solutions and Proofs . . . 47
4.3.1 Solution 01 . . . 47
4.3.2 Solution 02 . . . 49
4.3.3 Solution 03 . . . 50
4.3.4 Solution 04 . . . 51
4.3.5 Solution 05 . . . 52

5 References **53**
5.1 Bibliography . . . 53
5.2 Web Resources . . . 53

6 Final Words **54**

1 Basic Concepts of Logic

1.1 Comprehensive Course Summary

1.1.1 1. Fundamental Logical Connectives

Understanding how statements combine is crucial. Let P and Q be two propositions.

Key Logical Equivalences

- **Implication ($P \implies Q$):** By definition, $P \implies Q$ is logically equivalent to $\neg P \vee Q$.
- **Contrapositive (Contraposée):**

$$(P \implies Q) \iff (\neg Q \implies \neg P)$$

Key Note: If proving $P \implies Q$ is difficult, try proving $\neg Q \implies \neg P$.

- **Equivalence ($P \iff Q$):** To prove equivalence, you must usually prove two implications:

$$(P \implies Q) \wedge (Q \implies P)$$

1.1.2 2. The Art of Quantifiers (Deep Understanding)

The order of quantifiers changes the meaning of a sentence entirely.

Order Matters!

- $\forall x \in E, \exists y \in F$ **(Dependence):** Here, y depends on x. For every new x you pick, you can find a specific y_x.
  Example: Every student has a student ID.
- $\exists y \in F, \forall x \in E$ **(Uniformity):** Here, y is fixed and works for ALL x. This is a much stronger statement.
  Example: There is a Dean who manages every student.

1.1.3 3. Standard Proof Strategies (Methodology)

In L1 exams, identifying *which* method to use is 50% of the solution.

- **Direct Proof:** Assume P is true, use definitions/theorems to reach Q.
- **Proof by Contraposition (Par Contraposée):** Used when the hypothesis P is complicated or the conclusion Q involves a negation (e.g., $x \neq 0$).
  Method: Assume $\neg Q$ is true, show that $\neg P$ must be true.
- **Proof by Contradiction (Par l'Absurde):** Used when direct proof is stuck.
  Method: Assume the proposition is False (i.e., assume $P \wedge \neg Q$). Show that this leads to a mathematical absurdity (like $1 = 0$ or $x < x$).
- **Proof by Counter-Example (Contre-exemple):** **Only** used to disprove a Universal statement ($\forall$).
  Method: To show $(\forall x \in E, P(x))$ is False, just find ONE element x_0 where $P(x_0)$ fails.
- **Proof by Cases (Disjonction des cas):** Used when the problem behaves differently in different regions (e.g., $|x|$, n is even/odd).
  Method: Prove it for Case 1, then Case 2... ensure all cases cover the whole set.

1.1.4 4. Negation Rules (De Morgan for Quantifiers)

Must-Memorize Negation Formulas

1. $\neg(\forall x \in E, P(x)) \iff \exists x \in E, \neg P(x)$
2. $\neg(\exists x \in E, P(x)) \iff \forall x \in E, \neg P(x)$
3. $\neg(P \implies Q) \iff P \wedge \neg Q$ (**Critical! Note that $\implies$ disappears**)
4. $\neg(P \wedge Q) \iff \neg P \vee \neg Q$
5. $\neg(P \vee Q) \iff \neg P \wedge \neg Q$

1.2 List of Exercises

1.2.1 Exercise 01

Exercise 1.1

Which of the following propositions are true, which are false, and why?

(1) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R} : x + y > 4$

(2) $\exists x \in \mathbb{R}, \forall y \in \mathbb{R} : x + y > 4$

(3) $\forall x \in \mathbb{R}, \forall y \in \mathbb{R} : 2x + y > 0$

(4) $\exists x \in \mathbb{R}, \exists y \in \mathbb{R} : 2x + y > 0$

(5) $\exists x \in \mathbb{R}, \forall y \in \mathbb{R} : y^2 > x$

(6) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R} : (2x + y > 0 \vee 2x + y = 0)$

(7) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R} : (2x + y > 0 \wedge 2x + y = 0)$

(8) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R} : 2x + y > 0$

(9) $\exists x \in \mathbb{R}, \forall y \in \mathbb{R} : |x - y| > 0$

(10) $\exists x \in \mathbb{R}, \forall y \in \mathbb{R} : |2x + y| \leq 0$

→ Go to Proof

1.2.2 Exercise 02

Exercise 1.2

Write the negation of the following propositions:

(1) $\exists x \in \mathbb{R}, \forall y \in \mathbb{R} : 2x + y > 0$

(2) $\forall x \in \mathbb{R} : f(x) \geq 0$

(3) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R} : 2x + y > 0$

(4) $\exists x \in \mathbb{R}, \forall y \in \mathbb{R} : \sqrt{x + y} > 0$

(5) $\forall x \in \mathbb{R}, \forall y \in \mathbb{R} : \sqrt[5]{(x^2 + 1)} < 0$

(6) For $x \in \mathbb{R}, -(x^2 + 1) \leq 0 \implies x \in \mathbb{R}$

(7) $\forall \alpha \in \mathbb{R}, \exists \beta \in \mathbb{R}, \forall (x, y) \in I^2, |x - y| \leq \alpha \implies |f(x) - f(y)| \leq \beta$

(8) $\exists t \in \mathbb{R} : \forall x \in \mathbb{R} : f(t) \geq 0$

→ Go to Proof

1.2.3 Exercise 03

Exercise 1.3

Find the real numbers x that satisfy the following proposition:

$$\forall y \in [0,1] : x \geq y \implies x \geq 2y$$

→ Go to Proof

1.2.4 Exercise 04

Exercise 1.4

Let f, g be two functions from $\mathbb{R}$ to $\mathbb{R}$. Put the following expressions in terms of quantifiers:

(1) f is bounded above (majorée).

(2) f is bounded (bornée).

(3) f is even (paire).

(4) f is odd (impaire).

(5) f never vanishes (f ne s'annule jamais).

(6) f is periodic (périodique).

(7) f is increasing (croissante).

(8) f is strictly decreasing (strictement décroissante).

(9) f is not the zero function (n'est pas la fonction nulle).

(10) f is less than g (inférieure à g).

(11) f is not less than g.

→ Go to Proof

1.2.5 Exercise 05

Exercise 1.5

Prove the following using the specified method:

a) **Show by Contrapositive:** For $n \geq 2, n \in \mathbb{N}$:

$$\text{if } (n^2 - 1) \text{ is not divisible by 8, then } n \text{ is even.}$$

b) **Show by Absurdity:**

1) $\sqrt{2} \notin \mathbb{Q}$.
2) $\forall n \in \mathbb{N} : n^2$ is even $\implies n$ is even.

c) **Show by Counter-example that the following are false:**

1) (n is even) $\implies$ ($n^2 + 1$ is even).
2) $\forall x \in \mathbb{R}, \forall y \in \mathbb{R} : x + y > 0$.

→ Go to Proof

1.3 Detailed Solutions and Proofs

1.3.1 Solution 01

Rigorous Proof & Solution

Methodology:

- To prove a proposition is **True**, we must demonstrate it using definitions.
- To prove a proposition is **False**, we usually prove that its negation is True (often by finding a counter-example).

← Back to Exercise

(1) **Proposition:** $\forall x \in \mathbb{R}, \exists y \in \mathbb{R} : x + y > 4$.

- **Status: TRUE**.
- **Justification:** Let $x \in \mathbb{R}$ be arbitrary. We need to find a y such that $y > 4 - x$. Let us choose $y = 5 - x$. Since $x \in \mathbb{R}$, $5 - x \in \mathbb{R}$. Substituting into the inequality:

$$x + (5 - x) = 5$$

Since $5 > 4$, the condition holds.

(2) **Proposition:** $\exists x \in \mathbb{R}, \forall y \in \mathbb{R} : x + y > 4$.

- **Status: FALSE**.
- **Justification:** Let's look at the negation: $\forall x \in \mathbb{R}, \exists y \in \mathbb{R} : x + y \leq 4$. Let $x \in \mathbb{R}$ be arbitrary. We need to find y such that $y \leq 4 - x$. Let us choose $y = 4 - x - 1 = 3 - x$. Then $x + y = x + 3 - x = 3$. Since $3 \leq 4$, the negation is true. Thus, the original proposition is false.

(3) **Proposition:** $\forall x \in \mathbb{R}, \forall y \in \mathbb{R} : 2x + y > 0$.

- **Status: FALSE**.
- **Justification:** We provide a **counter-example**. Let $x = -1$ and $y = 0$. Then $2(-1) + 0 = -2$. Since $-2 \not> 0$, the statement is false.

(4) **Proposition:** $\exists x \in \mathbb{R}, \exists y \in \mathbb{R} : 2x + y > 0$.

- **Status: TRUE**.
- **Justification:** We just need one example. Let $x = 1$ and $y = 1$. $2(1)+1 = 3 > 0$. The statement is valid.

(5) **Proposition:** $\exists x \in \mathbb{R}, \forall y \in \mathbb{R} : y^2 > x$.

- **Status: TRUE**.
- **Justification:** We need to find a specific x that is strictly smaller than any real square. We know that $\forall y \in \mathbb{R}, y^2 \geq 0$. Let us choose $x = -1$. Then for all $y \in \mathbb{R}$, $y^2 \geq 0 > -1$. Thus $y^2 > x$ holds for all y.

(6) **Proposition:** $\forall x \in \mathbb{R}, \exists y \in \mathbb{R} : (2x + y > 0 \vee 2x + y = 0)$.

- **Status: TRUE**.
- **Justification:** The condition $(A > 0 \vee A = 0)$ is equivalent to $A \geq 0$. So we check: $\forall x, \exists y : 2x + y \geq 0$. Let $x \in \mathbb{R}$. Choose $y = -2x + 1$. Then $2x + (-2x + 1) = 1$. Since $1 \geq 0$, the condition holds.

(7) **Proposition:** $\forall x \in \mathbb{R}, \exists y \in \mathbb{R} : (2x + y > 0 \wedge 2x + y = 0)$.

- **Status: FALSE**.
- **Justification:** The condition requires a number to be strictly positive AND equal to zero simultaneously. This is logically impossible ($a > 0 \wedge a = 0$ is always False). Therefore, no such y exists.

(8) **Proposition:** $\forall x \in \mathbb{R}, \exists y \in \mathbb{R} : 2x + y > 0$.

- **Status: TRUE**.
- **Justification:** Let $x \in \mathbb{R}$. We need $y > -2x$. Choose $y = -2x + 1$. Then $2x + y = 1 > 0$. Valid.

(9) **Proposition:** $\exists x \in \mathbb{R}, \forall y \in \mathbb{R} : |x - y| > 0$.

- **Status: FALSE**.
- **Justification:** The negation is $\forall x \in \mathbb{R}, \exists y \in \mathbb{R} : |x - y| \leq 0$. Since absolute value is always non-negative, $|x - y| \leq 0 \implies |x - y| = 0 \implies x = y$. Let's prove the negation: For any x, does there exist a y such that $x = y$? Yes, choose $y = x$. Then $|x - x| = 0$, so $|x - y| > 0$ is false. Since the negation is True, the proposition is False.

(10) **Proposition:** $\exists x \in \mathbb{R}, \forall y \in \mathbb{R} : |2x + y| \leq 0$.

- **Status: FALSE**.
- **Justification:** $|A| \leq 0 \implies A = 0$. The proposition claims there exists an x such that for ALL y, $2x + y = 0$. This implies $y = -2x$. But y cannot be equal to a constant $-2x$ for all values of $y \in \mathbb{R}$. Counter-example for the "exists x": Let x be any candidate. Choose $y = -2x + 1$. Then $|2x + y| = |1| = 1 \not\leq 0$.

Rigorous Proof & Solution

1.3.2 Solution 02

Reminder on Negation Rules:

- $\forall$ becomes $\exists$.
- $\exists$ becomes $\forall$.
- $>$ becomes $\leq$, and $\geq$ becomes $<$.
- Negation of implication: $\neg(P \implies Q) \equiv P \wedge \neg Q$.

← Back to Exercise

(1) **Original:** $\exists x \in \mathbb{R}, \forall y \in \mathbb{R} : 2x + y > 0$

Negation: $\forall x \in \mathbb{R}, \exists y \in \mathbb{R} : 2x + y \leq 0$

(2) **Original:** $\forall x \in \mathbb{R} : f(x) \geq 0$

Negation: $\exists x \in \mathbb{R} : f(x) < 0$

(3) **Original:** $\forall x \in \mathbb{R}, \exists y \in \mathbb{R} : 2x + y > 0$

Negation: $\exists x \in \mathbb{R}, \forall y \in \mathbb{R} : 2x + y \leq 0$

(4) **Original:** $\exists x \in \mathbb{R}, \forall y \in \mathbb{R} : \sqrt{x + y} > 0$

Negation: $\forall x \in \mathbb{R}, \exists y \in \mathbb{R} : \sqrt{x + y} \leq 0$
(Note: Since $\sqrt{\cdot} \geq 0$, this effectively means $\sqrt{x + y} = 0$ or $x + y$ is undefined/negative context dependent, but mathematically strictly ≤ 0).

(5) **Original:** $\forall x \in \mathbb{R}, \forall y \in \mathbb{R} : \sqrt[5]{(x^2 + 1)} < 0$

Negation: $\exists x \in \mathbb{R}, \exists y \in \mathbb{R} : \sqrt[5]{(x^2 + 1)} \geq 0$

(6) **Original:** For $x \in \mathbb{R}, -(x^2 + 1) \leq 0 \implies x \in \mathbb{R}$

Note: This is of the form $P(x) \implies Q(x)$. The quantifier seems implicit $(\forall x)$. Assuming $\forall x \in \mathbb{R}$:

Negation: $\exists x \in \mathbb{R} : (-(x^2 + 1) \leq 0) \wedge (x \notin \mathbb{R})$
(This proposition is logically strange because we start with $x \in \mathbb{R}$, so $x \notin \mathbb{R}$ is impossible, making the negation False, hence the original statement is True tautologically).

(7) **Original:** $\forall \alpha \in \mathbb{R}, \exists \beta \in \mathbb{R}, \forall (x, y) \in I^2, |x - y| \leq \alpha \implies |f(x) - f(y)| \leq \beta$

Negation: Apply rules step by step:

- $\forall \alpha \to \exists \alpha$
- $\exists \beta \to \forall \beta$
- $\forall (x, y) \to \exists (x, y)$
- $P \implies Q \to P \wedge \neg Q$

Result: $\exists \alpha \in \mathbb{R}, \forall \beta \in \mathbb{R}, \exists (x, y) \in I^2 : |x - y| \leq \alpha \wedge |f(x) - f(y)| > \beta$

(8) **Original:** $\exists t \in \mathbb{R} : \forall x \in \mathbb{R} : f(t) \geq 0$

Negation: $\forall t \in \mathbb{R} : \exists x \in \mathbb{R} : f(t) < 0$
(Note: In the original statement, x does not appear in the predicate $f(t) \geq 0$. This is a "dummy quantifier" situation. The negation simply states that for any t, $f(t)$ is negative).

1.3.3 Solution 03

Rigorous Proof & Solution

Goal: Find $x \in \mathbb{R}$ such that $P(x)$ is true, where $P(x) : \forall y \in [0,1], (x \geq y \implies x \geq 2y)$.

← Back to Exercise

Recall that an implication $A \implies B$ is true unless A is true and B is false. So, we are looking for x where we **never** have ($x \geq y$ AND $x < 2y$) for any $y \in [0,1]$.
Case 1: $x < 0$ For any $y \in [0,1]$, $y \geq 0$. Since $x < 0$, the condition $x \geq y$ is always **False**. Since the hypothesis of the implication is False, the implication is automatically **True**. $\implies$ All $x < 0$ are solutions.
Case 2: $x \geq 0$ Here, the hypothesis $x \geq y$ can be true. We need to ensure that whenever $x \geq y$, we also have $x \geq 2y$.

- If $x \geq 2$, then since $y \in [0,1]$, the maximum value of $2y$ is 2. So $2y \leq 2 \leq x$, which means $x \geq 2y$ is always true. Thus, the implication holds for all $x \geq 2$.
- If $0 \leq x < 2$: Consider $y = x$ (if $x \leq 1$) or $y = 1$ (if $x > 1$). Let's test $x = 1$. Let $y = 1$. $1 \geq 1$ (True) $\implies$ $1 \geq 2(1)$ (False). The implication fails. Generally, if we choose $y = x/1.5$ (assuming it's in $[0,1]$), we satisfy $x \geq y$ but fail $x \geq 2y$. However, the only non-negative value where this implication holds for ALL $y \in [0,1]$ is $x = 0$. Check $x = 0$: $\forall y \in [0,1], 0 \geq y \implies y = 0$. Then $0 \geq 2(0)$ is True.

Conclusion: The set of real numbers satisfying the proposition is:

$$S =]-\infty, 0] \cup [2, +\infty[$$

1.3.4 Solution 04

Rigorous Proof & Solution

Translating Mathematical Sentences to Quantifiers:

← Back to Exercise

(1) f **is bounded above:**

$$\exists M \in \mathbb{R}, \forall x \in \mathbb{R}, f(x) \leq M$$

(2) f **is bounded:** (Bounded above AND bounded below)

$$\exists M \in \mathbb{R}^+, \forall x \in \mathbb{R}, |f(x)| \leq M$$

(3) f **is even:**

$$\forall x \in \mathbb{R}, f(-x) = f(x)$$

(4) f **is odd:**

$$\forall x \in \mathbb{R}, f(-x) = -f(x)$$

(5) f **never vanishes ($f(x) \neq 0$ everywhere):**

$$\forall x \in \mathbb{R}, f(x) \neq 0$$

(6) f **is periodic:**

$$\exists T \in \mathbb{R}^*, \forall x \in \mathbb{R}, f(x+T) = f(x)$$

(7) f **is increasing:**

$$\forall x_1, x_2 \in \mathbb{R}, x_1 \leq x_2 \implies f(x_1) \leq f(x_2)$$

(8) f **is strictly decreasing:**

$$\forall x_1, x_2 \in \mathbb{R}, x_1 < x_2 \implies f(x_1) > f(x_2)$$

(9) f **is not the zero function:** (Negation of $\forall x, f(x) = 0$)

$$\exists x \in \mathbb{R}, f(x) \neq 0$$

(10) f **is less than** g**:**

$$\forall x \in \mathbb{R}, f(x) \leq g(x)$$

(11) f **is not less than** g**:** (Negation of the above)

$$\exists x \in \mathbb{R}, f(x) > g(x)$$

1.3.5 Solution 05

Rigorous Proof & Solution

Proof Techniques Application:

← Back to Exercise

a) Contrapositive: *Statement:* $P \implies Q$ where P : (n^2-1) not divisible by 8 and Q : n is even. *Contrapositive:* $\neg Q \implies \neg P$. **To prove:** If n is odd, then (n^2-1) is divisible by 8.
Proof: Let n be an odd integer ($n \geq 3$ as $n \geq 2$). We can write $n = 2k+1$ for some $k \in \mathbb{N}^*$.

$$n^2 - 1 = (2k+1)^2 - 1 = 4k^2 + 4k + 1 - 1 = 4k^2 + 4k$$

Factorize:

$$n^2 - 1 = 4k(k+1)$$

Notice that $k(k+1)$ is the product of two consecutive integers. One of them must be even. So, $k(k+1) = 2m$ for some integer m. Substitute back:

$$n^2 - 1 = 4(2m) = 8m$$

Thus, $n^2 - 1$ is a multiple of 8. By contrapositive, the original statement is True. ■

b) Absurdity (Contradiction):
1) Prove $\sqrt{2} \notin \mathbb{Q}$. Assume the opposite: $\sqrt{2} \in \mathbb{Q}$. Then $\sqrt{2} = \frac{p}{q}$ with $p, q \in \mathbb{N}^*$ and $\gcd(p,q) = 1$ (fraction is irreducible). Square both sides: $2 = \frac{p^2}{q^2} \implies p^2 = 2q^2$. This implies p^2 is even, so p must be even. Let $p = 2k$. Substitute back: $(2k)^2 = 2q^2 \implies 4k^2 = 2q^2 \implies 2k^2 = q^2$. This implies q^2 is even, so q must be even. Contradiction: We assumed $\gcd(p,q) = 1$, but we found both are divisible by 2. Therefore, $\sqrt{2} \notin \mathbb{Q}$. ■
2) Prove n^2 even $\implies$ n even. Assume n^2 is even AND n is odd. $n = 2k+1$. $n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$. This form $2(\dots) + 1$ means n^2 is odd. Contradiction: We assumed n^2 is even. Thus, n must be even. ■

c) Counter-example:
1) (n even) $\implies$ ($n^2 + 1$ even). Let $n = 2$ (which is even). Check conclusion: $n^2 + 1 = 2^2 + 1 = 5$. 5 is odd, not even. The proposition is False.
2) $\forall x \in \mathbb{R}, \forall y \in \mathbb{R} : x + y > 0$. We need one pair (x, y) where sum is ≤ 0. Let $x = -2$ and $y = 1$. $x + y = -1$. $-1 \not> 0$. The proposition is False.

2 Sets and Mappings

2.1 Comprehensive Course Summary

Before solving Exercises 10 and 11, it is essential to review the definitions and the standard methods for constructing proofs in Set Theory.

2.1.1 1. Proof Strategies (Methodology)

1. **Proving Set Equality ($A = B$):** To prove that two sets are equal, you must show double inclusion:
$$A = B \iff (A \subset B \text{ AND } B \subset A)$$
2. **Proving Inclusion ($A \subset B$):** Pick an arbitrary element x in A and use logic/definitions to show that x must also be in B.
$$x \in A \implies \cdots \implies x \in B$$
3. **Proving Equivalence ($P \iff Q$):** You must prove two implications:
   - ($\Rightarrow$) Assume P is true, show Q is true.
   - ($\Leftarrow$) Assume Q is true, show P is true.
4. **Handling Direct Images ($y \in f(A)$):** By definition, to say $y \in f(A)$ means **there exists** an $x \in A$ such that $f(x) = y$.
$$y \in f(A) \iff \exists x \in A, f(x) = y$$
5. **Handling Inverse Images ($x \in f^{-1}(B)$):** This is often simpler. You just check the condition on the output.
$$x \in f^{-1}(B) \iff f(x) \in B$$

2.1.2 2. Cheat Sheet: Definitions & Operations

Set Operations

Let A and B be subsets of a set E.

- **Difference:** $A \setminus B = \{x \in E \mid x \in A \wedge x \notin B\} = A \cap B^c$
- **Symmetric Difference:** $A \Delta B = (A \setminus B) \cup (B \setminus A)$.
  Note: Elements belong to A or B, but **not both**.
- **De Morgan's Laws:**
$$(A \cup B)^c = A^c \cap B^c \quad \text{and} \quad (A \cap B)^c = A^c \cup B^c$$

Mappings (Direct vs Inverse)

Let $f : E \to F$ be a mapping. Let $A \subset E$ and $B \subset F$.

- **Direct Image $f(A)$:** The set of all outputs generated by inputs in A.
$$f(A) = \{f(x) \mid x \in A\}$$
- **Inverse Image $f^{-1}(B)$:** The set of all inputs that map to somewhere in B.
$$f^{-1}(B) = \{x \in E \mid f(x) \in B\}$$

2.1.3 3. Important Notes for Proofs

- **Warning on Intersections:** For direct images, the intersection is **not** preserved by equality, only by inclusion.
$$f(A_1 \cap A_2) \subset f(A_1) \cap f(A_2)$$
Equality holds ($=$) only if f is **injective**.
- **Inverse Images are "Nice":** The inverse image function f^{-1} preserves all set operations (Union, Intersection, Difference, Complement).
$$f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$$
- **Logic translation:** When proving sets, translate symbols to logic words:
  - $\cap$ becomes "AND" ($\wedge$)
  - $\cup$ becomes "OR" ($\vee$)
  - $\setminus$ becomes "AND NOT"

2.2 List of Exercises

2.2.1 Exercise 01

Exercise 2.1

Let A, B, C be three subsets of a set E. Show that:

a) $A \cap B = A \Leftrightarrow A \subset B$

b) $A \cup B = A \Leftrightarrow B \subset A$

c) $(A \cap C) \setminus (B \cap C) = (A \setminus B) \cap C$

d) $(A \Delta B) \cap C = (A \cap C) \Delta (B \cap C)$
Note: $A \Delta B = (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$

e) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

f) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

→ Go to Proof

2.2.2 Exercise 02

Exercise 2.2

Let $E = [0, 1]$, $F = [-1, 1]$, $G = [0, 2]$ be three intervals of $\mathbb{R}$. Consider the function $f : E \to G$ defined by $f(x) = x - 2$ and $g : E \to G$ defined by $g(x) = x^2 + 1$.

1. Determine: $f(1/2)$, $f(\{0\})$, $g([-1, 1])$, $g^{-1}([0, 2])$.
2. Is the function f bijective? Justify.
3. Is the function g bijective? Justify.

→ Go to Proof

2.2.3 Exercise 03

Exercise 2.3

Let $\mathcal{P}(E)$ be the power set of a set E. Show that:

1. $C_E(\bigcap_{i \in I} A_i) = \bigcup_{i \in I} C_E A_i$ (Generalized De Morgan)
2. $C_E(\bigcup_{i \in I} A_i) = \bigcap_{i \in I} C_E A_i$ (Generalized De Morgan)
3. $(\bigcup_{i \in I} A_i) \cap B = \bigcup_{i \in I} (A_i \cap B)$ (Distributivity)
4. $(\bigcap_{i \in I} A_i) \cup B = \bigcap_{i \in I} (A_i \cup B)$ (Distributivity)

→ Go to Proof

2.2.4 Exercise 04

Exercise 2.4

Are the following functions injective? Surjective? Bijective?

1. $f_1 : \mathbb{Z} \to \mathbb{Z}, \quad n \mapsto 2n$
2. $f_2 : \mathbb{Z} \to \mathbb{Z}, \quad n \mapsto -n$
3. $f_3 : \mathbb{R} \to \mathbb{R}, \quad x \mapsto x^2$
4. $f_4 : \mathbb{R} \to \mathbb{R}^+, \quad x \mapsto x^2$
5. $f_5 : \mathbb{C} \to \mathbb{C}, \quad z \mapsto z^2$
6. $f_6 : \mathbb{R}^2 \to \mathbb{R}^2, \quad (x, y) \mapsto (x + y, xy)$

→ Go to Proof

2.2.5 Exercise 05

Exercise 2.5

Let $g : [0, +\infty[\to [0, 1[$ be defined by:

$$g(x) = \frac{x}{x+1}$$

Prove that g is bijective and determine its inverse g^{-1}.

→ Go to Proof

2.2.6 Exercise 06

Exercise 2.6

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by: $f(x) = \frac{2x}{x^2+1}$

1. Is f injective? Surjective?
2. Show that $f(\mathbb{R}) = [-1, 1]$.
3. Show that the restriction $g : [-1, 1] \to [-1, 1]$ defined by $g(x) = f(x)$ is a bijection.

→ Go to Proof

2.2.7 Exercise 07

Exercise 2.7

Consider the mapping $f : \mathbb{R} \to \mathbb{R}$; $f(x) = E(x) = [x]$ (The Floor Function).

1. Is f injective? Is f surjective?
2. Find: $f^{-1}(\{1\})$, $f^{-1}(]0, 2[)$, $f^{-1}(]0, 2])$.
3. Find: $f(\mathbb{R})$, $f(]0, 2])$ and determine $f \circ f$.

→ Go to Proof

2.2.8 Exercise 08

Exercise 2.8

Let $f : E \to F$ and $g : F \to G$ be two mappings. Show that:

1. f and g injective $\implies$ $g \circ f$ injective.
2. f and g surjective $\implies$ $g \circ f$ surjective.
3. $g \circ f$ injective $\implies$ f injective.
4. $g \circ f$ surjective $\implies$ g surjective.

→ Go to Proof

2.2.9 Exercise 09

Exercise 2.9

Let E and F be two sets and $f : E \to F$. Let A and B be two subsets of F. Prove that:

1. $A \subset B \implies f^{-1}(A) \subset f^{-1}(B)$. Is the converse true?
2. $f^{-1}(A \cap B) \subset f^{-1}(A) \cap f^{-1}(B)$.
3. $f^{-1}(A \cup B) \subset f^{-1}(A) \cup f^{-1}(B)$.

→ Go to Proof

2.2.10 Exercise 10

Exercise 2.10

Let A, B, C be three subsets of a set E. Show that:

1. $A \cap B = A \iff A \subset B$
2. $A \cup B = A \iff B \subset A$
3. $(A \cap C) \setminus (B \cap C) = (A \setminus B) \cap C$
4. $(A \Delta B) \cap C = (A \cap C) \Delta (B \cap C)$

Note: $A \setminus B = \{x \in A, x \notin B\}$ and $A \Delta B = (A \setminus B) \cup (B \setminus A)$.

→ Go to Proof

2.2.11 Exercise 11

Exercise 2.11

Let $f : E \to F$ be a mapping and let $A_1, A_2 \in \mathcal{P}(E)$ and $B_1, B_2 \in \mathcal{P}(F)$. Show that:

1. $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$
2. $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$
3. $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$
4. $f(A_1 \cap A_2) \subset f(A_1) \cap f(A_2)$

→ Go to Proof

2.3 Detailed Solutions and Proofs

2.3.1 Solution 01

Rigorous Proof & Solution

Exercise 1 Solution: Set Operations

← Back to Exercise

a) Show that $A \cap B = A \Leftrightarrow A \subset B$
This is a logical equivalence ($\Leftrightarrow$). We must prove both implications.
Direction ($\Rightarrow$): Assume $A \cap B = A$. We must show $A \subset B$.

- Let $x \in A$.
- Since $A = A \cap B$, then $x \in A \cap B$.
- By definition of intersection ($x \in A \wedge x \in B$), this implies $x \in B$.
- Since $x \in A \implies x \in B$, we conclude $A \subset B$.

Direction ($\Leftarrow$): Assume $A \subset B$. We must show $A \cap B = A$.

- Inclusion ($\subset$): By definition, $A \cap B \subset A$ is always true for any sets.
- Inclusion ($\supset$): Let $x \in A$. Since we assumed $A \subset B$, then $x \in B$.
- Therefore, $x \in A$ and $x \in B$, which means $x \in A \cap B$.
- Thus $A \subset A \cap B$.
- Since $A \cap B \subset A$ and $A \subset A \cap B$, we conclude $A \cap B = A$.

Conclusion: The equivalence holds. ■

b) Show that $A \cup B = A \Leftrightarrow B \subset A$
Direction ($\Rightarrow$): Assume $A \cup B = A$.

- We know $B \subset A \cup B$ (basic property of union).
- Since $A \cup B = A$, substituting gives $B \subset A$.

Direction ($\Leftarrow$): Assume $B \subset A$.

- Always $A \subset A \cup B$.
- Conversely, let $x \in A \cup B$. Then $x \in A$ or $x \in B$.
- If $x \in A$, clearly $x \in A$.
- If $x \in B$, since $B \subset A$, then $x \in A$.
- In both cases, $x \in A$. Thus $A \cup B \subset A$.
- Therefore $A \cup B = A$.

Conclusion: The equivalence holds. ■

c) Show $(A \cap C) \setminus (B \cap C) = (A \setminus B) \cap C$

Let us transform the Left Hand Side (LHS) using definitions. Recall definition: $X \setminus Y = X \cap Y^c$ (where Y^c is the complement).

$$\begin{aligned} x \in (A \cap C) \setminus (B \cap C) &\iff x \in (A \cap C) \wedge x \notin (B \cap C) \\ &\iff (x \in A \wedge x \in C) \wedge \neg(x \in B \wedge x \in C) \\ &\iff (x \in A \wedge x \in C) \wedge (x \notin B \vee x \notin C) \quad \text{(De Morgan Law)} \\ &\iff [(x \in A \wedge x \in C) \wedge x \notin B] \vee [(x \in A \wedge x \in C) \wedge x \notin C] \end{aligned}$$

Notice the second part of the disjunction: $(x \in A \wedge x \in C \wedge x \notin C)$. This is impossible (False), as x cannot be in C and not in C. So the expression reduces to the first part:

$$\begin{aligned} &\iff (x \in A \wedge x \in C) \wedge x \notin B \\ &\iff (x \in A \wedge x \notin B) \wedge x \in C \\ &\iff x \in (A \setminus B) \wedge x \in C \\ &\iff x \in (A \setminus B) \cap C \end{aligned}$$

Conclusion: LHS = RHS. ■

d) Show $(A\Delta B) \cap C = (A \cap C)\Delta(B \cap C)$
Recall $X\Delta Y = (X \cap Y^c) \cup (Y \cap X^c)$. Let's expand the Right Hand Side (RHS), let $A' = A \cap C$ and $B' = B \cap C$.
$RHS = (A' \setminus B') \cup (B' \setminus A')$.
From part (c), we proved $(X \cap C) \setminus (Y \cap C) = (X \setminus Y) \cap C$. Therefore:

- $A' \setminus B' = (A \cap C) \setminus (B \cap C) = (A \setminus B) \cap C$
- $B' \setminus A' = (B \cap C) \setminus (A \cap C) = (B \setminus A) \cap C$

Substitute these back into the RHS expression:

$$\begin{aligned} RHS &= [(A \setminus B) \cap C] \cup [(B \setminus A) \cap C] \\ &= [(A \setminus B) \cup (B \setminus A)] \cap C \quad \text{(Distributivity)} \\ &= (A\Delta B) \cap C \\ &= LHS \end{aligned}$$

Conclusion: Equality holds. ■

e) Show $A \times (B \cap C) = (A \times B) \cap (A \times C)$

$$\begin{aligned} (x, y) \in A \times (B \cap C) &\iff x \in A \wedge y \in (B \cap C) \\ &\iff x \in A \wedge (y \in B \wedge y \in C) \\ &\iff (x \in A \wedge y \in B) \wedge (x \in A \wedge y \in C) \\ &\iff (x, y) \in (A \times B) \wedge (x, y) \in (A \times C) \\ &\iff (x, y) \in (A \times B) \cap (A \times C) \end{aligned}$$

Conclusion: Equality holds. ■

2.3.2 Solution 02

Rigorous Proof & Solution

Exercise 2 Solution: Functions on Intervals

← Back to Exercise

Data: $E = [0,1], F = [-1,1], G = [0,2]$. $f : E \to G, f(x) = x-2$. $g : E \to G, g(x) = x^2+1$.

1) Calculations

- $f(1/2) = (1/2) - 2 = -1.5$.
- $f(\{0\}) = \{f(0)\} = \{0-2\} = \{-2\}$.

  Remark: Note that $-1.5 \notin G$ and $-2 \notin G$. This implies the function f as defined in the exercise text is not well-defined for the codomain $G = [0,2]$. However, we proceed with calculations algebraically.
- $g([-1,1])$: Note that the domain of g is $E = [0,1]$, so we can only compute the image of the intersection $[-1,1] \cap E = [0,1]$. If the question assumes we extend domain, $x^2 \in [0,1]$ for $x \in [-1,1]$. $g([-1,1]) = \{x^2+1 \mid x \in [-1,1]\} = [0+1, 1+1] = [1,2]$.
- $g^{-1}([0,2]) = \{x \in E \mid g(x) \in [0,2]\}$. We need $0 \le x^2 + 1 \le 2$. Subtract 1: $-1 \le x^2 \le 1$. Since $x^2 \ge 0$ always, we look for $x^2 \le 1$, which means $x \in [-1,1]$. Intersecting with the domain $E = [0,1]$: $g^{-1}([0,2]) = [0,1]$.

2) Is f bijective? A function is bijective if it is both injective and surjective.
Surjectivity Check: Target set is $G = [0,2]$. Range of f on domain $E = [0,1]$: $0 \le x \le 1 \implies -2 \le x - 2 \le -1$. The image is $f(E) = [-2,-1]$. The intersection of the image $[-2,-1]$ and the codomain $G = [0,2]$ is empty. Therefore, no element $y \in G$ has a pre-image in E. f is **not surjective**.
Conclusion: f is **not bijective**.

3) Is g bijective? $g : [0,1] \to [0,2]$ with $g(x) = x^2 + 1$.
Injectivity: Let $x_1, x_2 \in [0,1]$. Assume $g(x_1) = g(x_2)$. $x_1^2 + 1 = x_2^2 + 1 \implies x_1^2 = x_2^2$. Since $x_1, x_2 \ge 0$, taking the square root gives $x_1 = x_2$. So, g is **injective**.
Surjectivity: Codomain is $G = [0,2]$. Image of domain $E = [0,1]$: $0 \le x \le 1 \implies 0 \le x^2 \le 1 \implies 1 \le x^2 + 1 \le 2$. Image is $[1,2]$. The interval $[0,1)$ is in the codomain G but not in the image. For example, take $y = 0 \in G$. Is there $x \in E$ such that $x^2 + 1 = 0$? $x^2 = -1$. No real solution. So g is **not surjective**.
Conclusion: g is **not bijective**.

2.3.3 Solution 03

Rigorous Proof & Solution

Exercise 3 Solution: Generalized Set Properties

← Back to Exercise

Let $C_E A$ denote the complement of A in E (often written A^c).
1) Proof of $C_E(\bigcap_{i\in I} A_i) = \bigcup_{i\in I} C_E A_i$
Let $x \in E$.

$$\begin{aligned} x \in C_E(\bigcap_{i\in I} A_i) &\iff x \notin \bigcap_{i\in I} A_i \\ &\iff \neg(\forall i \in I, x \in A_i) \\ &\iff \exists i \in I, x \notin A_i \\ &\iff \exists i \in I, x \in C_E A_i \\ &\iff x \in \bigcup_{i\in I} C_E A_i \end{aligned}$$

Therefore, the sets are equal. ■

2) Proof of $C_E(\bigcup_{i\in I} A_i) = \bigcap_{i\in I} C_E A_i$

$$\begin{aligned} x \in C_E(\bigcup_{i\in I} A_i) &\iff x \notin \bigcup_{i\in I} A_i \\ &\iff \neg(\exists i \in I, x \in A_i) \\ &\iff \forall i \in I, x \notin A_i \\ &\iff \forall i \in I, x \in C_E A_i \\ &\iff x \in \bigcap_{i\in I} C_E A_i \end{aligned}$$

Therefore, the sets are equal. ■

3) Proof of $(\bigcup_{i\in I} A_i) \cap B = \bigcup_{i\in I}(A_i \cap B)$

$$\begin{aligned} x \in (\bigcup_{i\in I} A_i) \cap B &\iff x \in \bigcup_{i\in I} A_i \wedge x \in B \\ &\iff (\exists i \in I, x \in A_i) \wedge x \in B \\ &\iff \exists i \in I, (x \in A_i \wedge x \in B) \\ &\iff \exists i \in I, x \in (A_i \cap B) \\ &\iff x \in \bigcup_{i\in I}(A_i \cap B) \end{aligned}$$

Therefore, the sets are equal. ■

Rigorous Proof & Solution

2.3.4 Solution 04

Exercise 4 Solution: Injectivity, Surjectivity, Bijectivity

← Back to Exercise

1) $f_1 : \mathbb{Z} \to \mathbb{Z}, \quad n \mapsto 2n$

- **Injective:** Let $f(n) = f(m) \implies 2n = 2m \implies n = m$. (Yes)
- **Surjective:** Let $y \in \mathbb{Z}$. Can we solve $2n = y$? Only if y is even. If $y = 1$, $2n = 1 \implies n = 1/2 \notin \mathbb{Z}$. (No)
- **Bijective:** No.

2) $f_2 : \mathbb{Z} \to \mathbb{Z}, \quad n \mapsto -n$

- **Injective:** $-n = -m \implies n = m$. (Yes)
- **Surjective:** For any $y \in \mathbb{Z}$, let $n = -y$. Then $f(n) = -(-y) = y$. Since $-y \in \mathbb{Z}$, a pre-image exists. (Yes)
- **Bijective:** Yes.

3) $f_3 : \mathbb{R} \to \mathbb{R}, \quad x \mapsto x^2$

- **Injective:** $f(-1) = 1$ and $f(1) = 1$. Since $-1 \neq 1$, not injective. (No)
- **Surjective:** Range of x^2 is $[0, +\infty[$. Codomain is $\mathbb{R}$. Negative numbers (e.g., -1) have no pre-image. (No)
- **Bijective:** No.

4) $f_4 : \mathbb{R} \to \mathbb{R}^+, \quad x \mapsto x^2$ (Assuming $\mathbb{R}^+ = [0, +\infty[$)

- **Injective:** Same as f_3, $f(-1) = f(1)$. (No)
- **Surjective:** Let $y \in \mathbb{R}^+$ (so $y \geq 0$). Let $x = \sqrt{y}$. Since $y \geq 0$, $\sqrt{y} \in \mathbb{R}$. $f(\sqrt{y}) = (\sqrt{y})^2 = y$. (Yes)
- **Bijective:** No.

5) $f_5 : \mathbb{C} \to \mathbb{C}, \quad z \mapsto z^2$

- **Injective:** $f(1) = 1$, $f(-1) = 1$. Distinct elements map to same image. (No)
- **Surjective:** For any complex number $w \in \mathbb{C}$, the equation $z^2 = w$ always has solutions in $\mathbb{C}$ (Fundamental Theorem of Algebra, or using polar form). (Yes)
- **Bijective:** No.

6) $f_6 : \mathbb{R}^2 \to \mathbb{R}^2, \quad (x, y) \mapsto (x + y, xy)$ Let $u = x + y$ and $v = xy$.

- **Injective:** Note that $(1, 2) \mapsto (3, 2)$ and $(2, 1) \mapsto (3, 2)$. Since $(1, 2) \neq (2, 1)$, it is not injective. (No)
- **Surjective:** Given $(u, v) \in \mathbb{R}^2$, do there exist real x, y such that $x + y = u$ and $xy = v$? x and y are roots of the quadratic equation $T^2 - uT + v = 0$. For real roots to exist, the discriminant $\Delta = u^2 - 4v$ must be ≥ 0. If we choose a target point $(u, v) = (0, 1)$, then $\Delta = 0 - 4 = -4 < 0$. No real solution exists. Thus $(0, 1)$ has no pre-image. (No)
- **Bijective:** No.

2.3.5 Solution 05

Rigorous Proof & Solution

Exercise 5 Solution: Bijectivity of a Rational Function

← Back to Exercise

Given: $g : [0, +\infty[\to [0, 1[$, with $g(x) = \frac{x}{x+1}$.
1. Injectivity: Let $x_1, x_2 \in [0, +\infty[$. Assume $g(x_1) = g(x_2)$.

$$\begin{aligned} \frac{x_1}{x_1+1} &= \frac{x_2}{x_2+1} \\ x_1(x_2+1) &= x_2(x_1+1) \\ x_1x_2 + x_1 &= x_2x_1 + x_2 \\ x_1 &= x_2 \end{aligned}$$

Thus, g is **injective**.
2. Surjectivity: Let $y \in [0, 1[$. We seek $x \in [0, +\infty[$ such that $g(x) = y$.

$$\begin{aligned} \frac{x}{x+1} &= y \\ x &= y(x+1) \\ x &= yx + y \\ x - yx &= y \\ x(1-y) &= y \\ x &= \frac{y}{1-y} \end{aligned}$$

Since $y \in [0, 1[$, we have $y \geq 0$ and $1 - y > 0$. Therefore, $x = \frac{y}{1-y} \geq 0$. So $x \in [0, +\infty[$.
Thus, g is **surjective**.
Conclusion: Since g is injective and surjective, it is **bijective**. The inverse function is:

$$g^{-1} : [0, 1[\to [0, +\infty[, \quad g^{-1}(y) = \frac{y}{1-y}$$

2.3.6 Solution 06

Rigorous Proof & Solution

Exercise 6 Solution: Study of $f(x) = \frac{2x}{x^2+1}$

← Back to Exercise

1) Injectivity and Surjectivity

- **Injectivity:** Check if $f(x) = f(y) \implies x = y$.

$$\begin{aligned} \frac{2x}{x^2+1} &= \frac{2y}{y^2+1} \\ 2x(y^2+1) &= 2y(x^2+1) \\ xy^2 + x &= x^2y + y \\ xy^2 - x^2y + x - y &= 0 \\ xy(y-x) - (y-x) &= 0 \\ (y-x)(xy-1) &= 0 \end{aligned}$$

  This implies $y = x$ OR $xy = 1$. Since $xy = 1$ is a possible solution different from $x = y$ (e.g., $x = 2, y = 1/2$), f is **not injective**.

- **Surjectivity:** Codomain is $\mathbb{R}$. Does $f(x) = 2$ have a solution? $\frac{2x}{x^2+1} = 2 \implies 2x = 2x^2 + 2 \implies 2x^2 - 2x + 2 = 0$. Discriminant $\Delta = 4 - 16 = -12 < 0$. No real solution. Thus, f is **not surjective**.

2) Show $f(\mathbb{R}) = [-1, 1]$ We want to find the set of all y such that $y = \frac{2x}{x^2+1}$ has a solution. $yx^2 - 2x + y = 0$. This is a quadratic in x. For $x \in \mathbb{R}$, we need $\Delta \geq 0$. $\Delta = (-2)^2 - 4(y)(y) = 4 - 4y^2$. $4 - 4y^2 \geq 0 \iff 1 - y^2 \geq 0 \iff y^2 \leq 1 \iff -1 \leq y \leq 1$. Thus, the image set is $[-1, 1]$.

3) Bijectivity of restriction $g : [-1, 1] \to [-1, 1]$

- **Surjectivity:** From part (2), the image of $\mathbb{R}$ is $[-1, 1]$. We must verify that we can reach all values using only $x \in [-1, 1]$. For any $y \in [-1, 1] \setminus \{0\}$, the equation $yx^2 - 2x + y = 0$ gives $x = \frac{2 \pm \sqrt{4-4y^2}}{2y} = \frac{1 \pm \sqrt{1-y^2}}{y}$. The product of roots is $x_1 x_2 = 1$. This means if one root is > 1, the other is < 1. So for every y, there exists at least one root x such that $|x| \leq 1$. Thus, g is **surjective**.

- **Injectivity:** Recall the condition for $f(x) = f(y)$ was $(y - x)(xy - 1) = 0$. So $x = y$ or $xy = 1$. If $x, y \in [-1, 1]$, then $|x| \leq 1$ and $|y| \leq 1$. The condition $xy = 1$ implies $|x||y| = 1$. Since both are ≤ 1, this forces $|x| = 1$ and $|y| = 1$. Possibilities: 1. $x = 1, y = 1 \implies x = y$. 2. $x = -1, y = -1 \implies x = y$. 3. $x = 1, y = -1 \implies xy = -1 \neq 1$. 4. $x = -1, y = 1 \implies xy = -1 \neq 1$. Therefore, in the domain $[-1, 1]$, $xy = 1$ implies $x = y$. Thus, g is **injective**.

Conclusion: g is bijective.

2.3.7 Solution 07

Rigorous Proof & Solution

Exercise 7 Solution: Floor Function

← Back to Exercise

$f(x) = \lfloor x \rfloor$ (largest integer $\leq x$).
1) Injectivity and Surjectivity

- **Injective:** No. $f(1.5) = 1$ and $f(1.9) = 1$. Distinct inputs map to the same output.
- **Surjective:** No. The codomain is $\mathbb{R}$, but the image is $\mathbb{Z}$. For example, $y = 0.5$ has no pre-image.

2) Inverse Images

- $f^{-1}(\{1\}) = \{x \in \mathbb{R} \mid \lfloor x \rfloor = 1\} = [1, 2[$.
- $f^{-1}(]0, 2[)$: We need x such that $\lfloor x \rfloor \in]0, 2[$. Since $\lfloor x \rfloor$ is an integer, the only integer in $]0, 2[$ is 1. So this is equivalent to $f^{-1}(\{1\}) = [1, 2[$.
- $f^{-1}(]0, 2])$: Integers in $]0, 2]$ are $\{1, 2\}$. So we need $\lfloor x \rfloor = 1$ or $\lfloor x \rfloor = 2$. $x \in [1, 2[\cup[2, 3[= [1, 3[$.

3) Images and Composition

- $f(\mathbb{R}) = \mathbb{Z}$ (The set of all integers).
- $f(]0, 2])$: We look at the set of values $\lfloor x \rfloor$ takes for $x \in]0, 2]$. If $0 < x < 1$, $\lfloor x \rfloor = 0$. If $1 \leq x < 2$, $\lfloor x \rfloor = 1$. If $x = 2$, $\lfloor x \rfloor = 2$. So the image is $\{0, 1, 2\}$.
- $f \circ f$: $(f \circ f)(x) = f(f(x)) = \lfloor \lfloor x \rfloor \rfloor$. Let $n = \lfloor x \rfloor$. By definition $n \in \mathbb{Z}$. The floor of an integer is the integer itself. So $\lfloor n \rfloor = n$. Therefore, $(f \circ f)(x) = \lfloor x \rfloor = f(x)$.

2.3.8 Solution 08

Rigorous Proof & Solution

Exercise 8 Solution: Composition Properties

← Back to Exercise

1) f, g **injective** $\implies$ $g \circ f$ **injective** Assume $g(f(x)) = g(f(y))$. Since g is injective, $f(x) = f(y)$. Since f is injective, $x = y$. Therefore, $g \circ f$ is injective. ■
2) f, g **surjective** $\implies$ $g \circ f$ **surjective** Let $z \in G$. Since g is surjective, $\exists y \in F$ such that $g(y) = z$. Since f is surjective, for this y, $\exists x \in E$ such that $f(x) = y$. Substitute y: $g(f(x)) = z$. Thus $\exists x \in E, (g \circ f)(x) = z$. Therefore, $g \circ f$ is surjective. ■
3) $g \circ f$ **injective** $\implies$ f **injective** Assume $f(x_1) = f(x_2)$. Apply g to both sides: $g(f(x_1)) = g(f(x_2))$. This means $(g \circ f)(x_1) = (g \circ f)(x_2)$. Since $g \circ f$ is assumed injective, we conclude $x_1 = x_2$. Thus f is injective. ■
4) $g \circ f$ **surjective** $\implies$ g **surjective** Let $z \in G$. Since $g \circ f$ is surjective, $\exists x \in E$ such that $(g \circ f)(x) = z$. Let $y = f(x)$. Then $y \in F$. We have $g(y) = g(f(x)) = z$. So for any $z \in G$, we found a $y \in F$ such that $g(y) = z$. Thus g is surjective. ■

2.3.9 Solution 09

Rigorous Proof & Solution

Exercise 9 Solution: Inverse Image Properties

← Back to Exercise

1) $A \subset B \implies f^{-1}(A) \subset f^{-1}(B)$ Let $x \in f^{-1}(A)$. By definition, $f(x) \in A$. Since $A \subset B$, then $f(x) \in B$. By definition of inverse image, $x \in f^{-1}(B)$. Thus $f^{-1}(A) \subset f^{-1}(B)$. ■
Is the converse true? No. Counter-example: Let $E = \{1\}$, $F = \{a, b\}$. Let $f(1) = a$. Let $A = \{b\}$ and $B = \emptyset$. $f^{-1}(A) = \emptyset$ and $f^{-1}(B) = \emptyset$. $\emptyset \subset \emptyset$ is true. But $\{b\} \not\subset \emptyset$. So converse is false.
2) $f^{-1}(A \cap B) \subset f^{-1}(A) \cap f^{-1}(B)$ Let $x \in f^{-1}(A \cap B)$. $\iff f(x) \in A \cap B$. $\iff f(x) \in A \wedge f(x) \in B$. $\iff x \in f^{-1}(A) \wedge x \in f^{-1}(B)$. $\iff x \in f^{-1}(A) \cap f^{-1}(B)$. (Note: Since steps are equivalences, equality holds, which implies inclusion). ■
3) $f^{-1}(A \cup B) \subset f^{-1}(A) \cup f^{-1}(B)$ Let $x \in f^{-1}(A \cup B)$. $\iff f(x) \in A \cup B$. $\iff f(x) \in A \vee f(x) \in B$. $\iff x \in f^{-1}(A) \vee x \in f^{-1}(B)$. $\iff x \in f^{-1}(A) \cup f^{-1}(B)$. (Again, equality holds). ■

2.3.10 Solution 10

Rigorous Proof & Solution

Exercise 10 Solution: Set Operations Proofs

← Back to Exercise

1) Show that $A \cap B = A \iff A \subset B$

- ($\Rightarrow$) Assume $A \cap B = A$. We know that for any sets, $(A \cap B) \subset B$. Substituting the assumption, we get $A \subset B$.
- ($\Leftarrow$) Assume $A \subset B$. We know $(A \cap B) \subset A$ is always true. Now, let $x \in A$. Since $A \subset B$, then $x \in B$. Since $x \in A$ and $x \in B$, then $x \in A \cap B$. Thus $A \subset (A \cap B)$. Conclusion: $A = A \cap B$.

2) Show that $A \cup B = A \iff B \subset A$

- ($\Rightarrow$) Assume $A \cup B = A$. We know $B \subset (A \cup B)$. Substituting the assumption, we get $B \subset A$.
- ($\Leftarrow$) Assume $B \subset A$. We know $A \subset (A \cup B)$ is always true. Now, let $x \in (A \cup B)$. Then $x \in A$ or $x \in B$. Since $B \subset A$, if $x \in B \implies x \in A$. In both cases, $x \in A$. Thus $(A \cup B) \subset A$. Conclusion: $A \cup B = A$.

3) Show $(A \cap C) \setminus (B \cap C) = (A \setminus B) \cap C$ Recall definition: $X \setminus Y = X \cap Y^c$.

$$\begin{aligned} LHS &= (A \cap C) \cap (B \cap C)^c \\ &= (A \cap C) \cap (B^c \cup C^c) \quad \text{(De Morgan's Law)} \\ &= ((A \cap C) \cap B^c) \cup ((A \cap C) \cap C^c) \quad \text{(Distributivity)} \\ &= (A \cap B^c \cap C) \cup (A \cap \underbrace{(C \cap C^c)}_{\emptyset}) \\ &= (A \cap B^c \cap C) \cup \emptyset \\ &= (A \cap B^c) \cap C \\ &= (A \setminus B) \cap C = RHS \quad \blacksquare \end{aligned}$$

4) Show $(A \Delta B) \cap C = (A \cap C) \Delta (B \cap C)$ Recall: $X \Delta Y = (X \setminus Y) \cup (Y \setminus X)$.

$$\begin{aligned} LHS &= [(A \setminus B) \cup (B \setminus A)] \cap C \\ &= [(A \setminus B) \cap C] \cup [(B \setminus A) \cap C] \quad \text{(Distributivity)} \end{aligned}$$

Using the result from question (3) above:

$$(A \setminus B) \cap C = (A \cap C) \setminus (B \cap C)$$

$$(B \setminus A) \cap C = (B \cap C) \setminus (A \cap C)$$

Substitute these back:

$$\begin{aligned} LHS &= [(A \cap C) \setminus (B \cap C)] \cup [(B \cap C) \setminus (A \cap C)] \\ &= (A \cap C) \Delta (B \cap C) = RHS \quad \blacksquare \end{aligned}$$

2.3.11 Solution 11

Rigorous Proof & Solution

Exercise 11 Solution: Image and Inverse Image

← Back to Exercise

1) Show $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$

- ($\subset$) Let $y \in f(A_1 \cup A_2)$. Then $\exists x \in A_1 \cup A_2$ such that $f(x) = y$. If $x \in A_1 \implies y \in f(A_1)$. If $x \in A_2 \implies y \in f(A_2)$. Thus $y \in f(A_1) \cup f(A_2)$.
- ($\supset$) Since $A_1 \subset A_1 \cup A_2$, then $f(A_1) \subset f(A_1 \cup A_2)$. Similarly $f(A_2) \subset f(A_1 \cup A_2)$. Therefore, $f(A_1) \cup f(A_2) \subset f(A_1 \cup A_2)$.

Equality holds. ■

2) Show $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$

$$\begin{aligned} x \in f^{-1}(B_1 \cup B_2) &\iff f(x) \in B_1 \cup B_2 \\ &\iff f(x) \in B_1 \vee f(x) \in B_2 \\ &\iff x \in f^{-1}(B_1) \vee x \in f^{-1}(B_2) \\ &\iff x \in f^{-1}(B_1) \cup f^{-1}(B_2) \quad \blacksquare \end{aligned}$$

3) Show $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$

$$\begin{aligned} x \in f^{-1}(B_1 \cap B_2) &\iff f(x) \in B_1 \cap B_2 \\ &\iff f(x) \in B_1 \wedge f(x) \in B_2 \\ &\iff x \in f^{-1}(B_1) \wedge x \in f^{-1}(B_2) \\ &\iff x \in f^{-1}(B_1) \cap f^{-1}(B_2) \quad \blacksquare \end{aligned}$$

4) Show $f(A_1 \cap A_2) \subset f(A_1) \cap f(A_2)$ Let $y \in f(A_1 \cap A_2)$. Then $\exists x \in A_1 \cap A_2$ such that $f(x) = y$. Since $x \in A_1 \cap A_2 \implies x \in A_1 \implies y = f(x) \in f(A_1)$. Since $x \in A_1 \cap A_2 \implies x \in A_2 \implies y = f(x) \in f(A_2)$. Therefore, $y \in f(A_1) \cap f(A_2)$.
Note: The reverse inclusion is not true in general unless f is injective. ■

3 Binary Relations on a Set

3.1 Comprehensive Course Summary

To solve exercises on Binary Relations effectively, you must master the four fundamental properties and know how to combine them.

3.1.1 1. The Four Fundamental Properties

Let $\mathcal{R}$ be a binary relation on a set E.

Definitions & Logic

1. **Reflexivity (Inflexive):** Every element is related to itself.

$$\forall x \in E, \quad x\mathcal{R}x$$

2. **Symmetry (Symétrique):** The direction does not matter.

$$\forall x, y \in E, \quad x\mathcal{R}y \implies y\mathcal{R}x$$

3. **Antisymmetry (Antisymétrique):** No loops allowed unless it is the same element.

$$\forall x, y \in E, \quad (x\mathcal{R}y \land y\mathcal{R}x) \implies x = y$$

4. **Transitivity (Transitive):** The relation bridges gaps.

$$\forall x, y, z \in E, \quad (x\mathcal{R}y \land y\mathcal{R}z) \implies x\mathcal{R}z$$

3.1.2 2. Types of Relations

Classification Map

- **Equivalence Relation:** Must be (1) Reflexive + (2) Symmetric + (4) Transitive.
  Key Concept: It partitions the set E into disjoint **Equivalence Classes**.
- **Partial Order (Ordre Partiel):** Must be (1) Reflexive + (3) Antisymmetric + (4) Transitive.
  Key Concept: Used to compare elements (like $\leq$ or $\subset$).
- **Total Order:** A partial order where **every pair is comparable**.

$$\forall x, y \in E, \quad (x\mathcal{R}y) \lor (y\mathcal{R}x)$$

3.1.3 3. How to Write Rigorous Proofs (Methodology)

Step-by-Step Proof Strategies

Strategy A: Proving Antisymmetry (Most Common Mistake)
Goal: Show that if elements relate both ways, they are identical.

1. **Start with:** "Let $x, y \in E$. Assume $x\mathcal{R}y$ AND $y\mathcal{R}x$."
2. **Write equations:** Translate relations into math (e.g., $x \leq y$ and $y \leq x$).
3. **Manipulate:** Combine the two facts to show $x - y = 0$ or similar.
4. **Conclude:** "Therefore, $x = y$."

Strategy B: Finding the Equivalence Class $\dot{x}$
Goal: Find the set of all y related to a fixed x.

1. **Definition:** Write $\dot{x} = \{y \in E \mid x\mathcal{R}y\}$.
2. **Set up Equation:** Write the condition for $x\mathcal{R}y$ (e.g., $x^2 - y^2 = 0$).
3. **Solve for y:** Treat x as a constant and solve for y.
4. **Result:** The solutions found constitute the set $\dot{x}$.

Strategy C: Total vs Partial Order
To check if an order is **Total**:

- Try to find a counter-example (two elements a, b where neither $a\mathcal{R}b$ nor $b\mathcal{R}a$ works).
- If you can't compare them (e.g., subsets $\{1\}$ and $\{2\}$), it is **Partial**.
- If you can always compare them (e.g., numbers in $\mathbb{R}$), it is **Total**.

3.2 List of Exercises

3.2.1 Exercise 01

Exercise 3.1

Part a) Let $\mathcal{R}$ be the relation defined on $\mathbb{R}$ by:

$$x\mathcal{R}y \iff x^2 - y^2 = x - y$$

1. Show that $\mathcal{R}$ is an equivalence relation.
2. Determine the equivalence class of x, denoted $\dot{x}$ or $Cl(x)$, for any real number x.

Part b) Let $\mathcal{R}$ be the relation defined on $I =]1, +\infty[$ by:

$$x\mathcal{R}y \iff \frac{x}{1+x^2} \geq \frac{y}{1+y^2}$$

1. Show that $\mathcal{R}$ is a partial order.
2. Is this order total?

→ Go to Proof

3.2.2 Exercise 02

Exercise 3.2

Are the relations $\mathcal{R}$ defined below **partial orders** on $\mathbb{R}$? Justify your answer.

1. $\forall x, y \in \mathbb{R}, \quad x\mathcal{R}y \iff x < y$
2. $\forall x, y \in \mathbb{R}, \quad x\mathcal{R}y \iff x \leq y$
3. $\forall x, y \in \mathbb{R}, \quad x\mathcal{R}y \iff e^x \leq e^y$
4. $\forall x, y \in \mathbb{R}, \quad x\mathcal{R}y \iff |x| \leq |y|$
5. $\forall x, y \in \mathbb{R}, \quad x\mathcal{R}y \iff x - y \in \mathbb{N}$
6. $\forall x, y \in \mathbb{R}, \quad x\mathcal{R}y \iff x - y \in \mathbb{Z}$

→ Go to Proof

3.2.3 Exercise 03

Exercise 3.3

Are the relations $\mathcal{R}$ defined below **equivalence relations** on $\mathbb{C}$? Justify your answer.

1. $\forall z, z' \in \mathbb{C}, \quad z\mathcal{R}z' \iff |z| \leq |z'|$
2. $\forall z, z' \in \mathbb{C}, \quad z\mathcal{R}z' \iff \frac{|z|}{|z'|} = 1$ (Assume $z, z' \neq 0$ for definition, or interpret as $|z| = |z'|$)
3. $\forall z, z' \in \mathbb{C}, \quad z\mathcal{R}z' \iff |z - z'| = 1$
4. $\forall z, z' \in \mathbb{C}, \quad z\mathcal{R}z' \iff e^{|z-z'|} = 1$

→ Go to Proof

3.2.4 Exercise 04

Exercise 3.4

One gives the relation marked $\prec$, defined on $\mathbb{R}^2$ by:

$$(x, y) \prec (x', y') \iff x \leq x' \wedge y \leq y'$$

1. Show that $\prec$ is a partial order relation on $\mathbb{R}^2$.
2. Is this order total?

→ Go to Proof

3.2.5 Exercise 05

Exercise 3.5

Let $\mathcal{R}$ be a relation defined on $E = \mathbb{Z} \times \mathbb{N}^*$ by:

$$(a, b)\mathcal{R}(a', b') \iff ab' = a'b$$

1. Show that $\mathcal{R}$ is an equivalence relation.
2. Let $(p, q) \in \mathbb{Z} \times \mathbb{N}^*$ with $p \wedge q = 1$ (meaning $\gcd(p, q) = 1$). Describe the equivalence class of (p, q).

→ Go to Proof

3.3 Detailed Solutions and Proofs

3.3.1 Solution 01

Rigorous Proof & Solution

Part a) The relation $x^2 - y^2 = x - y$

← Back to Exercise

Let us rewrite the defining condition.

$$x\mathcal{R}y \iff x^2 - x = y^2 - y$$

Let define the function $f : \mathbb{R} \to \mathbb{R}$ by $f(t) = t^2 - t$. Then the relation becomes: $x\mathcal{R}y \iff f(x) = f(y)$.

1. Proving $\mathcal{R}$ is an equivalence relation:

- **Reflexivity:** For any $x \in \mathbb{R}$, $f(x) = f(x)$ is trivially true. Thus $x\mathcal{R}x$.
- **Symmetry:** Let $x, y \in \mathbb{R}$ such that $x\mathcal{R}y$. This means $f(x) = f(y)$. Since equality is symmetric, $f(y) = f(x)$, which implies $y\mathcal{R}x$.
- **Transitivity:** Let $x, y, z \in \mathbb{R}$ such that $x\mathcal{R}y$ and $y\mathcal{R}z$. We have $f(x) = f(y)$ and $f(y) = f(z)$. By properties of equality, $f(x) = f(z)$. Thus $x\mathcal{R}z$.

Conclusion: $\mathcal{R}$ is an equivalence relation.

2. The Equivalence Class of x: By definition, $\dot{x} = \{y \in \mathbb{R} \mid x\mathcal{R}y\}$.

$$\begin{aligned} x\mathcal{R}y &\iff x^2 - y^2 = x - y \\ &\iff (x-y)(x+y) - (x-y) = 0 \\ &\iff (x-y)[(x+y) - 1] = 0 \\ &\iff x - y = 0 \quad \vee \quad x + y - 1 = 0 \\ &\iff y = x \quad \vee \quad y = 1 - x \end{aligned}$$

Conclusion: The equivalence class of x is the set $\dot{x} = \{x, 1 - x\}$.

Part b) The relation on $]1, +\infty[$: $x\mathcal{R}y \iff \frac{x}{1+x^2} \geq \frac{y}{1+y^2}$

Let $g(t) = \frac{t}{1+t^2}$. The relation is $x\mathcal{R}y \iff g(x) \geq g(y)$. First, let us analyze the monotonicity of g on $I =]1, +\infty[$. The derivative is:

$$g'(t) = \frac{1(1+t^2) - t(2t)}{(1+t^2)^2} = \frac{1 - t^2}{(1+t^2)^2}$$

Since $t > 1$, implies $t^2 > 1$, so $1 - t^2 < 0$. Thus, g is **strictly decreasing** on I. Consequently, g is injective: $g(x) = g(y) \implies x = y$.

1. Proving $\mathcal{R}$ is a partial order:

- **Reflexivity:** $\forall x \in I, g(x) \geq g(x)$ is true.
- **Antisymmetry:** Let $x, y \in I$ such that $x\mathcal{R}y$ and $y\mathcal{R}x$.

  $$(g(x) \geq g(y)) \wedge (g(y) \geq g(x)) \implies g(x) = g(y)$$

  Since g is strictly decreasing (and thus injective) on I, $g(x) = g(y) \implies x = y$. Thus, $\mathcal{R}$ is antisymmetric.

- **Transitivity:** Let $x, y, z \in I$. If $x\mathcal{R}y$ ($g(x) \geq g(y)$) and $y\mathcal{R}z$ ($g(y) \geq g(z)$), then by transitivity of $\geq$ in $\mathbb{R}$, we have $g(x) \geq g(z)$. Thus $x\mathcal{R}z$.

Conclusion: $\mathcal{R}$ is a partial order.
2. Is the order total? We need to check if $\forall x, y \in I, (x\mathcal{R}y \vee y\mathcal{R}x)$. Since $g(x), g(y) \in \mathbb{R}$ and the standard order $\geq$ on $\mathbb{R}$ is total, it is always true that either $g(x) \geq g(y)$ or $g(y) \geq g(x)$. Therefore, for any x, y, either $x\mathcal{R}y$ or $y\mathcal{R}x$ holds. **Conclusion:** Yes, it is a total order.

Rigorous Proof & Solution

3.3.2 Solution 02

Checking Partial Orders on $\mathbb{R}$

← Back to Exercise

Recall: A partial order must be Reflexive, Antisymmetric, and Transitive.

1. $x\mathcal{R}y \iff x < y$
   - **Reflexivity:** Is $x < x$? No.
   - **Conclusion:** Not a partial order (It is a strict order).
2. $x\mathcal{R}y \iff x \leq y$
   - This is the standard definition of order on $\mathbb{R}$. It satisfies all three axioms.
   - **Conclusion:** Yes, it is a partial order.
3. $x\mathcal{R}y \iff e^x \leq e^y$
   - Since the exponential function is strictly increasing, $e^x \leq e^y \iff x \leq y$.
   - This is equivalent to case (2).
   - **Conclusion:** Yes, it is a partial order.
4. $x\mathcal{R}y \iff |x| \leq |y|$
   - **Antisymmetry Check:** Let $x = 1$ and $y = -1$. $|1| \leq |-1|$ is $1 \leq 1$ (True). $|-1| \leq |1|$ is $1 \leq 1$ (True). However, $x \neq y$ ($1 \neq -1$).
   - **Conclusion:** Not a partial order (Antisymmetry fails).
5. $x\mathcal{R}y \iff x - y \in \mathbb{N}$
   - **Reflexivity:** $x - x = 0 \in \mathbb{N}$. (True).
   - **Antisymmetry:** Assume $x - y = n \in \mathbb{N}$ and $y - x = m \in \mathbb{N}$. Summing them: $(x-y)+(y-x) = n+m \implies 0 = n+m$. Since n, m are non-negative integers, their sum is 0 implies $n = 0$ and $m = 0$. Thus $x - y = 0 \implies x = y$. (True).
   - **Transitivity:** Assume $x - y = n \in \mathbb{N}$ and $y - z = m \in \mathbb{N}$. $x - z = (x - y) + (y - z) = n + m$. Since $\mathbb{N}$ is closed under addition, $n + m \in \mathbb{N}$. (True).
   - **Conclusion:** Yes, it is a partial order.
6. $x\mathcal{R}y \iff x - y \in \mathbb{Z}$
   - **Antisymmetry Check:** Let $x = 1.5$ and $y = 0.5$. $x - y = 1 \in \mathbb{Z}$ (So $x\mathcal{R}y$ is True). $y - x = -1 \in \mathbb{Z}$ (So $y\mathcal{R}x$ is True). But $x \neq y$.
   - **Conclusion:** Not a partial order (Antisymmetry fails).

3.3.3 Solution 03

Rigorous Proof & Solution

Checking Equivalence Relations on $\mathbb{C}$

← Back to Exercise

Recall: An equivalence relation must be Reflexive, Symmetric, and Transitive.

1. $z\mathcal{R}z' \iff |z| \leq |z'|$
   - **Symmetry Check:** Let $z = 1$ and $z' = 2$. $|1| \leq |2|$ is True. Does $|2| \leq |1|$ hold? No.
   - **Conclusion:** Not an equivalence relation.
2. $z\mathcal{R}z' \iff \frac{|z|}{|z'|} = 1$ (assuming $z, z' \neq 0$)
   - This condition is equivalent to $|z| = |z'|$.
   - **Reflexivity:** $|z| = |z|$ (True).
   - **Symmetry:** $|z| = |z'| \implies |z'| = |z|$ (True).
   - **Transitivity:** $|z| = |z'|$ and $|z'| = |z''| \implies |z| = |z''|$ (True).
   - **Conclusion:** Yes, it is an equivalence relation. (The equivalence classes are concentric circles centered at the origin).
3. $z\mathcal{R}z' \iff |z - z'| = 1$
   - **Reflexivity Check:** We need $|z - z| = 1$. But $|z - z| = |0| = 0 \neq 1$.
   - **Conclusion:** Not an equivalence relation (Reflexivity fails).
4. $z\mathcal{R}z' \iff e^{|z-z'|} = 1$
   - Let $d = |z - z'|$. Note that $d \in \mathbb{R}$ and $d \geq 0$. The equation is $e^d = 1$. In $\mathbb{R}$, the exponential function $e^x = 1$ has the unique solution $x = 0$. Therefore:

     $$e^{|z-z'|} = 1 \iff |z - z'| = 0 \iff z = z'$$

   - This relation is simply the identity relation ($=$).
   - Equality is always Reflexive, Symmetric, and Transitive.
   - **Conclusion:** Yes, it is an equivalence relation.

3.3.4 Solution 04

Rigorous Proof & Solution

Exercise 4 Solution: Product Order on $\mathbb{R}^2$

← Back to Exercise

Let $\prec$ be the relation on $\mathbb{R}^2$ defined by $(x, y) \prec (x', y') \iff (x \leq x' \wedge y \leq y')$.
1. Showing $\prec$ is a partial order We must verify Reflexivity, Antisymmetry, and Transitivity.

- **Reflexivity:** Let $u = (x, y) \in \mathbb{R}^2$. Since $\leq$ is reflexive in $\mathbb{R}$, we have $x \leq x$ and $y \leq y$. Therefore, $(x, y) \prec (x, y)$. The relation is reflexive.
- **Antisymmetry:** Let $u = (x, y)$ and $v = (x', y')$ in $\mathbb{R}^2$. Assume $u \prec v$ and $v \prec u$.

$$u \prec v \implies x \leq x' \wedge y \leq y'$$
$$v \prec u \implies x' \leq x \wedge y' \leq y$$

  From $x \leq x'$ and $x' \leq x$, we deduce $x = x'$. From $y \leq y'$ and $y' \leq y$, we deduce $y = y'$. Thus $(x, y) = (x', y')$, so $u = v$. The relation is antisymmetric.
- **Transitivity:** Let $u = (x, y), v = (x', y'), w = (x'', y'')$. Assume $u \prec v$ and $v \prec w$.

$$(x \leq x' \wedge y \leq y') \wedge (x' \leq x'' \wedge y' \leq y'')$$

  Using the transitivity of $\leq$ in $\mathbb{R}$: $x \leq x'$ and $x' \leq x'' \implies x \leq x''$. $y \leq y'$ and $y' \leq y'' \implies y \leq y''$. Therefore, $u \prec w$. The relation is transitive.

Conclusion: $\prec$ is a partial order on $\mathbb{R}^2$.
2. Is the order total? The order is total if $\forall u, v \in \mathbb{R}^2, (u \prec v \vee v \prec u)$. **Counter-example:** Let $u = (1, 5)$ and $v = (2, 3)$.

- Check $u \prec v$: $1 \leq 2$ (True) but $5 \leq 3$ (False). So $u \not\prec v$.
- Check $v \prec u$: $2 \leq 1$ (False). So $v \not\prec u$.

Since these two elements are not comparable, the order is **not total** (it is partial).

Rigorous Proof & Solution

3.3.5 Solution 05

Exercise 5 Solution: Construction of Rational Numbers

← Back to Exercise

Context: This relation allows constructing $\mathbb{Q}$ from $\mathbb{Z} \times \mathbb{N}^*$. Relation on $E = \mathbb{Z} \times \mathbb{N}^*$: $(a,b)\mathcal{R}(a',b') \iff ab' = a'b$.

1. Showing $\mathcal{R}$ is an equivalence relation

- **Reflexivity:** Let $(a,b) \in E$. We check if $(a,b)\mathcal{R}(a,b)$. The condition becomes $ab = ab$, which is trivially true in $\mathbb{Z}$.
- **Symmetry:** Let $(a,b),(a',b') \in E$ such that $(a,b)\mathcal{R}(a',b')$. By definition, $ab' = a'b$. This equality implies $a'b = ab'$ (symmetry of equality). This matches the definition of $(a',b')\mathcal{R}(a,b)$.
- **Transitivity:** Let $u = (a,b), v = (c,d), w = (e,f)$ elements of E. Assume $u\mathcal{R}v$ and $v\mathcal{R}w$.

$$ad = cb \tag{1}$$
$$cf = ed \tag{2}$$

We want to show $af = eb$. Multiply (1) by f and (2) by b:

$$adf = cbf$$
$$cfb = edb$$

Since multiplication is commutative, $cbf = cfb$. Thus:

$$adf = edb$$

$$d(af) = d(eb)$$

Since $d \in \mathbb{N}^*$, $d \neq 0$. We can simplify by d (cancellation law in integral domain $\mathbb{Z}$).

$$af = eb$$

This implies $(a,b)\mathcal{R}(e,f)$.

Conclusion: $\mathcal{R}$ is an equivalence relation.

2. Describe the equivalence class of (p,q) where $\gcd(p,q) = 1$ Let $Cl(p,q) = \{(x,y) \in \mathbb{Z} \times \mathbb{N}^* \mid (x,y)\mathcal{R}(p,q)\}$.

$$\begin{aligned}(x,y) \in Cl(p,q) &\iff xq = yp \\ &\iff \frac{x}{y} = \frac{p}{q} \quad \text{(Valid in } \mathbb{Q}\text{)}\end{aligned}$$

Since the fraction $\frac{p}{q}$ is irreducible ($\gcd(p,q) = 1$), any other pair (x,y) representing the same rational number must be a multiple of (p,q). There exists a $k \in \mathbb{Z}^*$ such that $x = kp$ and $y = kq$. However, the domain requires $y \in \mathbb{N}^*$. Since $q \in \mathbb{N}^*$, for $y = kq$ to be positive, k must be positive. Thus $k \in \mathbb{N}^*$.

Conclusion: The equivalence class is the set of all "unsimplified fractions" equal to p/q:

$$Cl(p,q) = \{(kp,kq) \mid k \in \mathbb{N}^*\}$$

4 Algebraic Structures

4.1 Comprehensive Course Summary

4.1.1 Algebraic Structures Summary

Key Logical Equivalences

1. **Groups (Les Groupes)** $(G, *)$
   - **Structure:** A set G with an internal binary operation $*$.
   - **Axioms:**
     1. Associativity: $\forall x, y, z \in G, (x * y) * z = x * (y * z)$.
     2. Neutral Element (e): $\exists e \in G, \forall x \in G, x * e = e * x = x$.
     3. Inverse: $\forall x \in G, \exists x^{-1} \in G, x * x^{-1} = x^{-1} * x = e$.
   - **Abelian Group:** If $x * y = y * x$ (Commutative).
   - **Subgroup ($H \leq G$):** $H \neq \emptyset$ and $\forall x, y \in H \implies xy^{-1} \in H$.

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2. **Rings (Les Anneaux)** $(A, +, \times)$
   - **Structure:** Set A with two operations $+$ and $\times$.
   - **Axioms:**
     1. $(A, +)$ is an Abelian Group (Neutral element usually 0_A).
     2. $\times$ is Associative.
     3. $\times$ is Distributive over $+$ ($a(b + c) = ab + ac$).
   - **Ideal (I):** Subgroup for $(+)$ AND absorbs multiplication ($\forall a \in A, \forall x \in I \implies ax \in I$).

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3. **Fields (Les Corps)** $(\mathbb{K}, +, \times)$
   - **Definition:** A commutative ring where every non-zero element is invertible.
   - **Structure:** $(\mathbb{K}, +)$ is an abelian group, and $(\mathbb{K}^*, \times)$ is an abelian group.
   - **Note:** A field has only two ideals: $\{0\}$ and $\mathbb{K}$ itself.

4.1.2 Keys to Solving Exercises (Moyens & Astuces)

Key Logical Equivalences

How to prove... / Comment montrer que...

- **It is a Subgroup (Sous-groupe):**
  1. Check if $e \in H$ (Non-empty).
  2. Check stability & inverse at once: Let $x, y \in H$, show $x * y^{-1} \in H$.
  3. *Finite Case:* If H is finite, just check stability ($xy \in H$).
- **It is an Ideal (Idéal):**
  1. $(I, +)$ is a subgroup (Check $x - y \in I$).
  2. Absorption: Take any $r \in Ring$ and any $x \in Ideal$. Show $r \cdot x \in I$.
- **Symmetric Group (S_n) Calculation:**
  - **Disjoint Cycles:** Always convert permutations to disjoint cycles first.
  - **Order:** The order of a permutation is the LCM (ppcm) of the lengths of its disjoint cycles.
  - **Signature (ε):** $\varepsilon(\sigma) = (-1)^{n-k}$ where n is total elements and k is number of cycles (including size 1).
- **Morphism (ϕ):**
  - To prove injectivity: Show $\ker(\phi) = \{e\}$.
  - To prove surjectivity: Show $\text{Im}(\phi) = \text{Codomain}$.

4.1.3 Visual Illustrations (Schémas)

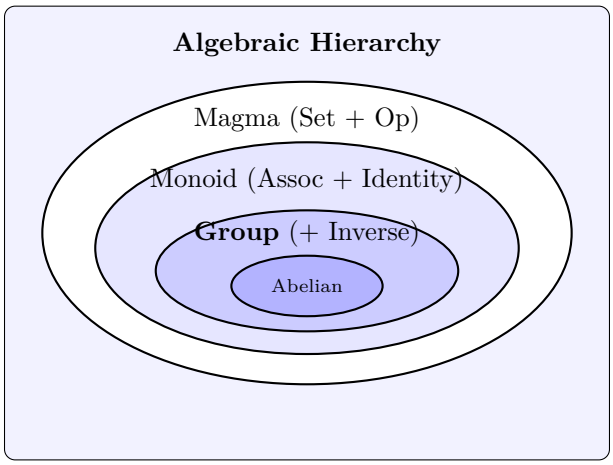

Figure 1: Hierarchy from Magma to Abelian Group

4.2 List of Exercises

4.2.1 Exercise 01

Exercise 4.1

Show that the following laws provide the indicated set G with a group structure, and specify whether it is Abelian (Commutative):

1. On the set $G =]-1, 1[$:
$$x * y = \frac{x+y}{1+xy}$$
2. On the set $G = \mathbb{R}^2$:
$$(x, y) * (x', y') = (x + x', ye^{x'} + y'e^{-x})$$

→ Go to Proof

4.2.2 Exercise 02

Exercise 4.2

Let G be a group defined as follows:

$$\forall x \in G, \quad x^2 = e$$

(where e is the neutral element of G).
Show that G is a commutative (Abelian) group.

→ Go to Proof

4.2.3 Exercise 03

Exercise 4.3

Let G be a group, and let $H \leq G$ and $K \leq G$ (meaning H and K are subgroups of G).

1. Show that: $H \cap K \leq G$.
2. Show that: $H \cup K \leq G \implies H \subseteq K$ or $K \subseteq H$.

→ Go to Proof

4.2.4 Exercise 04

Exercise 4.4

Calculate the conjugate aba^{-1} in the symmetric group for the following permutations:

1. $a = (132)(12)$ and $b = (13)$.
2. $a = (32)$ and $b = (123)$.

Note: Composition of permutations is performed from right to left, i.e., $f \circ g(x) = f(g(x))$.

→ Go to Proof

4.2.5 Exercise 05

Exercise 4.5

Let $(A, +, \times)$ be a commutative ring and let M be a subset of A. We call the **annihilator** (or annulator) of M the set:

$$Ann(M) = \{x \in A \mid \forall y \in M, x \cdot y = 0\}$$

Show that $Ann(M)$ is an **ideal** of the ring $(A, +, \times)$.

→ Go to Proof

4.3 Detailed Solutions and Proofs

4.3.1 Solution 01

Rigorous Proof & Solution

Exercise 1: Analysis of Group Structures

← Back to Exercise

Part 1: The law on $G =]-1,1[$ defined by $x * y = \frac{x+y}{1+xy}$

1. **Internal Composition Law (Closure):** Let $x, y \in]-1,1[$. We must show that $-1 < \frac{x+y}{1+xy} < 1$. Since $|x| < 1$ and $|y| < 1$, we have $x^2 < 1$ and $y^2 < 1$. Consider the expression $1 - (x * y)^2$:

$$1 - \left(\frac{x+y}{1+xy}\right)^2 = \frac{(1+xy)^2 - (x+y)^2}{(1+xy)^2} = \frac{1 + 2xy + x^2y^2 - (x^2 + 2xy + y^2)}{(1+xy)^2}$$

$$= \frac{1 - x^2 - y^2 + x^2y^2}{(1+xy)^2} = \frac{(1-x^2)(1-y^2)}{(1+xy)^2}$$

Since $1 - x^2 > 0$ and $1 - y^2 > 0$, the numerator is positive. The denominator is positive. Thus $1 - (x*y)^2 > 0 \implies (x*y)^2 < 1 \implies |x*y| < 1$. The law is internal.

2. **Associativity:** Compute $(x * y) * z$ and $x * (y * z)$.

$$(x * y) * z = \frac{\frac{x+y}{1+xy} + z}{1 + \frac{x+y}{1+xy}z} = \frac{x + y + z + xyz}{1 + xy + xz + yz}$$

Because the expression is symmetric in x, y, z, the order of operations does not matter. Thus, the law is associative.

3. **Neutral Element:** We seek $e \in G$ such that $x * e = x$.

$$\frac{x+e}{1+xe} = x \implies x + e = x + x^2e \implies e(1 - x^2) = 0$$

Since $x \in]-1,1[$, $1 - x^2 \neq 0$, so $e = 0$. Since $0 \in]-1,1[$, the neutral element exists and is $e = 0$.

4. **Inverse Element:** We seek y such that $x * y = 0$.

$$\frac{x+y}{1+xy} = 0 \implies x + y = 0 \implies y = -x$$

Since $|x| < 1 \implies |-x| < 1$, the inverse is in G.

5. **Commutativity:** $x * y = \frac{x+y}{1+xy} = \frac{y+x}{1+yx} = y * x$. The group is Abelian.

Conclusion 1: $(]-1,1[, *)$ is an Abelian Group.

Part 2: The law on $G = \mathbb{R}^2$ defined by $(x, y) * (x', y') = (x + x', ye^{x'} + y'e^{-x})$

1. **Internal Composition Law:** For $(x, y), (x', y') \in \mathbb{R}^2$, $x + x' \in \mathbb{R}$ and $ye^{x'} + y'e^{-x} \in \mathbb{R}$. Thus the operation is internal.

2. **Associativity:** Let $u = (x, y), v = (x', y'), w = (x'', y'')$.

$$\begin{aligned}(u * v) * w &= (x + x', ye^{x'} + y'e^{-x}) * (x'', y'') \\ &= \Big((x + x') + x'', (ye^{x'} + y'e^{-x})e^{x''} + y''e^{-(x+x')}\Big) \\ &= \Big(x + x' + x'', ye^{x'+x''} + y'e^{x''-x} + y''e^{-x-x'}\Big)\end{aligned}$$

   Now compute $u * (v * w)$:

$$\begin{aligned}u * (v * w) &= (x, y) * (x' + x'', y'e^{x''} + y''e^{-x'}) \\ &= \Big(x + (x' + x''), ye^{(x'+x'')} + (y'e^{x''} + y''e^{-x'})e^{-x}\Big) \\ &= \Big(x + x' + x'', ye^{x'+x''} + y'e^{x''-x} + y''e^{-x'-x}\Big)\end{aligned}$$

   The expressions are identical. The law is associative.

3. **Neutral Element:** Let $e_G = (e_1, e_2)$. We require $(x, y) * (e_1, e_2) = (x, y)$.

$$\begin{cases} x + e_1 = x \\ ye^{e_1} + e_2e^{-x} = y \end{cases} \implies \begin{cases} e_1 = 0 \\ y(1) + e_2e^{-x} = y \implies e_2 = 0 \end{cases}$$

   We verify left neutral: $(0, 0) * (x, y) = (0 + x, 0e^x + ye^{-0}) = (x, y)$. Thus $e_G = (0, 0)$.

4. **Inverse Element:** Let (x', y') be the inverse of (x, y). We need $(x, y)*(x', y') = (0, 0)$.

$$\begin{cases} x + x' = 0 \implies x' = -x \\ ye^{x'} + y'e^{-x} = 0 \end{cases}$$

   Substitute $x' = -x$:

$$ye^{-x} + y'e^{-x} = 0 \implies (y + y')e^{-x} = 0 \implies y' = -y$$

   The inverse is $(-x, -y)$.

5. **Commutativity:** $(x', y') * (x, y) = (x' + x, y'e^x + ye^{-x'})$. Compare the second component with the original: $ye^{x'} + y'e^{-x}$. These are generally different (e.g., if $x = 1, x' = 0, y = 1, y' = 0$, we get 1 vs 0). Not commutative.

Conclusion 2: $(\mathbb{R}^2, *)$ is a Non-Abelian Group.

4.3.2 Solution 02

Rigorous Proof & Solution

Exercise 2: Groups with $x^2 = e$

← Back to Exercise

Hypothesis: Let G be a group such that $\forall x \in G, x^2 = e$. This implies that for every element x, $x * x = e$. Multiplying by x^{-1} gives $x = x^{-1}$. **Every element is its own inverse.**
To Prove: G is commutative ($\forall x, y \in G, xy = yx$).
Proof: Let $x, y \in G$. Consider the element (xy). By the hypothesis, $(xy)^2 = e$. This means $(xy)(xy) = e$.
Multiply on the left by x and on the right by y:

$$x(xyxy)y = x(e)y$$

$$(xx)yx(yy) = xy$$

Since $xx = e$ and $yy = e$:

$$e(yx)e = xy$$

$$yx = xy$$

Since this holds for any $x, y \in G$, the group is commutative. ■

4.3.3 Solution 03

Rigorous Proof & Solution

Exercise 3: Intersection and Union of Subgroups

← Back to Exercise

1) Show that $H \cap K \leq G$ To prove a subset is a subgroup, we check: $e \in S$, closed under product, closed under inverse.

- **Identity:** Since $H \leq G$ and $K \leq G$, $e \in H$ and $e \in K$. Thus $e \in H \cap K$.
- **Closure:** Let $x, y \in H \cap K$. Then $x, y \in H \implies xy \in H$ (since H is subgroup). And $x, y \in K \implies xy \in K$ (since K is subgroup). Therefore, $xy \in H \cap K$.
- **Inverse:** Let $x \in H \cap K$. $x \in H \implies x^{-1} \in H$. $x \in K \implies x^{-1} \in K$. Therefore, $x^{-1} \in H \cap K$.

Conclusion: $H \cap K$ is a subgroup of G.

2) Show that $H \cup K \leq G \implies H \subseteq K \vee K \subseteq H$ We proceed by **contraposition** (or contradiction). Assume $H \not\subseteq K$ AND $K \not\subseteq H$. We will show that $H \cup K$ is NOT a subgroup.

- Since $H \not\subseteq K$, $\exists h \in H$ such that $h \notin K$.
- Since $K \not\subseteq H$, $\exists k \in K$ such that $k \notin H$.
- Clearly, $h \in H \cup K$ and $k \in H \cup K$.
- Consider the product $x = h * k$.

If $H \cup K$ were a subgroup, then x must belong to $H \cup K$. Thus $x \in H$ or $x \in K$.
Case 1: Assume $x \in H$. Since $h \in H$, then $h^{-1} \in H$. By closure of H, $h^{-1} * x \in H$. $h^{-1} * (h * k) = (h^{-1} * h) * k = e * k = k$. This implies $k \in H$, which contradicts our assumption that $k \notin H$.
Case 2: Assume $x \in K$. Since $k \in K$, then $k^{-1} \in K$. By closure of K, $x * k^{-1} \in K$. $(h * k) * k^{-1} = h * (k * k^{-1}) = h * e = h$. This implies $h \in K$, which contradicts our assumption that $h \notin K$.
Since $x = hk$ belongs to neither H nor K, it does not belong to $H \cup K$. Thus, $H \cup K$ is not closed under the operation, so it is not a subgroup.
Conclusion: For the union to be a subgroup, one must be contained in the other. ■

Rigorous Proof & Solution

4.3.4 Solution 04

Exercise 4: Conjugacy in Symmetric Groups

← Back to Exercise

Methodology: To calculate aba^{-1}, we can use the property of conjugation: if $b = (x_1x_2\dots x_k)$, then $aba^{-1} = (a(x_1)a(x_2)\dots a(x_k))$. Alternatively, we can compute the product directly (right to left).
1) Calculate aba^{-1} for $a = (132)(12)$ and $b = (13)$.
Step 1: Simplify a. a is a product of cycles. Let's find its disjoint cycle form. $a = (132)\circ(12)$.

- Image of 1: (12) sends $1 \to 2$. Then (132) sends $2 \to 1$. Result: $1 \to 1$.
- Image of 2: (12) sends $2 \to 1$. Then (132) sends $1 \to 3$. Result: $2 \to 3$.
- Image of 3: (12) fixes 3. Then (132) sends $3 \to 2$. Result: $3 \to 2$.

Thus, $a = (23)$ (The transposition swapping 2 and 3). Note that $a^{-1} = (23)$ as well (transpositions are self-inverse).
Step 2: Calculate aba^{-1}.

$$aba^{-1} = (23)(13)(23)$$

Using the conjugation property: apply a to the elements of $b = (13)$.

- $a(1) = 1$ (since 1 is not in the cycle (23)).
- $a(3) = 2$.

Therefore, $aba^{-1} = (a(1)a(3)) = (12)$.
Verification by direct product: Input: $\xrightarrow{(23)} \dots \xrightarrow{(13)} \dots \xrightarrow{(23)}$ Output

- $1 \xrightarrow{(23)} 1 \xrightarrow{(13)} 3 \xrightarrow{(23)} 2$. (So $1 \to 2$)
- $2 \xrightarrow{(23)} 3 \xrightarrow{(13)} 1 \xrightarrow{(23)} 1$. (So $2 \to 1$)
- $3 \xrightarrow{(23)} 2 \xrightarrow{(13)} 2 \xrightarrow{(23)} 3$. (So $3 \to 3$)

Result is indeed (12).

2) Calculate aba^{-1} for $a = (32)$ and $b = (123)$. Note that $a = (32) = (23)$.
Using the conjugation property: $b = (1, 2, 3)$. $aba^{-1} = (a(1), a(2), a(3))$.

- $a(1) = 1$ (fixed by (23)).
- $a(2) = 3$.
- $a(3) = 2$.

Thus, $aba^{-1} = (132)$.
Verification: (23)(123)(23).

- $1 \xrightarrow{a^{-1}} 1 \xrightarrow{b} 2 \xrightarrow{a} 3$. (So $1 \to 3$)
- $3 \xrightarrow{a^{-1}} 2 \xrightarrow{b} 3 \xrightarrow{a} 2$. (So $3 \to 2$)
- $2 \xrightarrow{a^{-1}} 3 \xrightarrow{b} 1 \xrightarrow{a} 1$. (So $2 \to 1$)

Cycle is (132). Matches.

4.3.5 Solution 05

Rigorous Proof & Solution

Exercise 5: Annihilator Ideal

← Back to Exercise

Definitions & Goal: We must prove that $I = Ann(M) = \{x \in A \mid \forall y \in M, xy = 0\}$ is an ideal of A. Since A is commutative, left ideals and right ideals are the same. We must check two conditions:

1. $(I, +)$ is a subgroup of $(A, +)$.
2. Absorption: $\forall a \in A, \forall x \in I, ax \in I$.

Proof:
1. Subgroup Test:

- **Non-empty:** Does $0_A \in I$? Yes, because $0_A \cdot y = 0_A$ for all $y \in M$ (property of the zero element in a ring).
- **Stability under subtraction:** Let $x_1, x_2 \in I$. This means $\forall y \in M, x_1 y = 0$ and $x_2 y = 0$. We calculate $(x_1 - x_2)y$:

$$(x_1 - x_2)y = x_1 y - x_2 y = 0 - 0 = 0$$

Since this holds for all $y \in M$, then $(x_1 - x_2) \in I$.

Thus, $(I, +)$ is a subgroup of A.
2. Absorption Property: Let $a \in A$ (any ring element) and $x \in I$ (an element of the annihilator). We want to show that $ax \in I$. By definition of $x \in I$, we know $xy = 0$ for all $y \in M$. Consider the product $(ax)y$ for any $y \in M$:

$$(ax)y = a(xy) \quad \text{(Associativity of ring multiplication)}$$

Since $xy = 0$:

$$a(xy) = a(0) = 0$$

Therefore, $(ax)y = 0$ for all $y \in M$. This implies $ax \in Ann(M)$.
Conclusion: Since $Ann(M)$ is an additive subgroup and absorbs multiplication by elements of A, it is an **ideal** of A. ■

5 References

5.1 Bibliography

Recommended Textbooks & References

The following references were consulted/recommended for the topics covered in this course (Logic, Sets, Relations, and Algebraic Structures):

References

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[5] Dummit, D. S., & Foote, R. M. (2004). *Abstract Algebra* (3rd ed.). Wiley. (Comprehensive reference for advanced algebraic concepts).

[6] Ayres, F., & Jaisingh, L. R. (2003). *Schaum's Outline of Abstract Algebra*. McGraw-Hill. (Contains numerous solved exercises similar to the ones in this document).

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5.2 Web Resources

- Wolfram MathWorld - Algebra Resources
- Wikipedia Mathematics Portal
- Algeria National Higher School of Mathematics
- Moodle platform of Echahid Cheikh Larbi Tebessi University.

6 Final Words

Summary

This document serves as a structured instructional guide and exercise collection for the foundational topics of Algebra 01. It is designed to support first-year university students in developing a solid understanding of core algebraic concepts, including sets and relations, functions, basic algebraic structures, and fundamental problem-solving techniques.
The material is organized in a progressive manner, beginning with essential definitions and theoretical principles, and advancing toward worked examples and carefully selected exercise series. Each chapter aims to strengthen mathematical reasoning by linking abstract concepts with practical applications. Through a balanced combination of theory, illustrative examples, and practice problems, this work seeks to enhance both conceptual clarity and analytical skills, providing learners with a reliable foundation for further studies in mathematics and related disciplines.

Last Word

This book reaches its conclusion after offering a clear and methodical pathway through the fundamental principles of Algebra 01. It was prepared with the intention of helping students build confidence in mathematical reasoning while developing the rigor and precision required for higher-level studies. By combining theoretical exposition with structured exercises, this work aims to bridge the gap between abstract algebraic concepts and their practical understanding.
We have reached the end of this series.
If this work has been helpful or if any benefit has been gained from it, its purpose has been fulfilled. It is sincerely hoped that the knowledge acquired here will support learners in their academic journey and future endeavors.
If you found this work helpful or derived any benefit from it, I ask for nothing in return but a sincere prayer for mercy and forgiveness for my parents.

Abdeljali Mostefaoui

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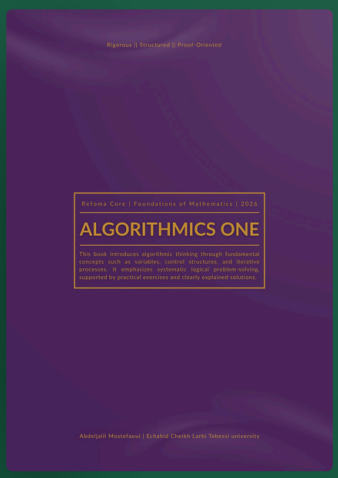
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